

MATHEMATICIANS: YESTERDAY AND TODAY*

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Besides teaching, part of the job at the university is to do research. In the case of mathematics, it is mathematical research. Mathematics has had a history of more than 2500 years, and it is interesting to note how the subject develops through the course of history.

Mathematics, indeed all sciences, was considered part of the general discipline known as natural philosophy. In Greek times, mathematicians were sometimes also philosophers, physicists, astronomers. Archimedes (287 - 212 B.C.), considered one of the greatest intellect of antiquity, was known to be a physicist of the first rank. In fact until recent times, very few leading mathematicians stayed exclusively within the bondary of mathematics as we know it today. We have among the illustrious examples people like Isaac Newton (1642 - 1727), one of the inventors of calculus and discoverer of the law of gravitation; and Carl Friedrich Gauss (1777 - 1855), great contributor to various branches of mathematics (number theory, complex variable theory, differential geometry, statistics etc.) and a first class astronomer.

On the other hand, it was not until the last two hundred years or so that the majority of creative mathematicians were associated with the universities. In those days there were a few academies (for example the Berlin and the St. Petersburg Academies) which were supported by royalty. Members of the academies were paid to produce scientific work.

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But then not every mathematician was associated with such institutions. We have in mind the case of Pierre Fermat (1601 - 1665). A jurist and counsellor to the King by profession, Fermat dabbled in mathematics as a hobby, but his enormous contributions to the subject place him eminently among the leaders of his time. He was, in addition to Newton and Leibnitz (1646 - 1716), a third inventor of the calculus. The many works he did in number theory virtually steered the direction in which later research in this field was to follow. To most people, Fermat is best known for his 'Last Theorem', which states that $x^n + y^n = z^n$, for $n > 2$, has no integral solutions in x, y, z . It is now an often told story that Fermat had scribbled in one of his books that he had found 'a truly marvelous proof of the theorem', but the margin in the book was too small for him to include his proof. After nearly three hundred years of concerted efforts - by professional and amateur mathematicians, and after several offers of attractive prizes for the first person to solve the problem, the 'Last Theorem' remains today among the 'first unsolved questions' in mathematics. History has shown that Fermat's problem is a classic instance in mathematics where pursuing a deceptively simple-looking statement could lead to a deep analysis and significant advancement of the subject concerned. We are of course thinking about the work of Ernst Kummer (1810 - 1893), who in his numerous attempts to solve the Fermat problem introduced the notion of ideal numbers, marking the beginning of the theory of algebraic numbers. It is now generally believed that Fermat's problem, if true, cannot be solved by the elementary methods that were invented since Greek times. Even Fermat's invention of the method of 'infinite descent' could only be used to solve some special cases of the problem. It follows that Fermat's 'proof' of his 'Last Theorem' was probably false.

Things took a gradual turn for mathematicians in the late eighteenth century. By now the majority of them were professors teaching or doing research at the universities or

academies. Thus for example Newton spent most of his creative life in the University of Cambridge, and Gauss was appointed Director of the Observatory at the University of Göttingen in Germany. After his death, Dirichlet (1805 - 1859) and later Riemann (1826 - 1866) succeeded the chair, and established a tradition of excellence in mathematics there and in all of Germany.

In France all the leading mathematicians were also associated with the universities or the French Academy - with the exception of Evariste Galois (1811 - 1832), whose great work on the theory of equations heralded the advent of group theory. Galois's case was of course one of the tragic instances in the history of mathematics. He had completed his revolutionary work at the age of 19, had sent copies of his manuscript to the French Academy but received no response. And one always recalls how regrettably his disappointments led to the ending of his life (by engaging in a duel) at the young age of 21. Luckily these things happen, when they do, only very rarely.

In any case, by the year 1850 it was common for professional mathematicians to seek teaching jobs at the universities (sometimes even in high schools when times were bad). These people could devote full time to their endeavours; the training of bright young students and the pursuit of mathematical research. Their contributions to society were recognised, and their achievements respected. In comparison with earlier periods, the nineteenth century saw a marked increase of mathematical activities and accomplishments. Most of the mathematics that are being taught today in the university were developed during that time.

By the beginning of the twentieth century, the University of Göttingen in Germany had won the distinction of being the centre of mathematics, largely under the able leader-

ship of Felix Klein (1849 - 1925) and David Hilbert (1862 - 1943). Aspiring and established mathematicians from Japan, Europe and America went on pilgrimages to Göttingen, to study or exchange ideas with the people they had come to meet. It was around this time - at the International Congress of Mathematicians in Paris in 1900 - that the famous 23 Hilbert problems were presented to mathematicians of the world. These important problems covered a wide range of subjects, and their status at each period since 1900 have often been used as a criterion to assess the progress of mathematics. As recent as 1974 an international symposium for this purpose was held in the U.S.A. The symposium, entitled 'Mathematical developments arising from Hilbert problems', attempted to survey those areas of importance in contemporary mathematical research which could be traced back to the 23 Hilbert problems. To give an idea of the magnitude and significance of the contributions of the Hilbert problems to the development of mathematics of our time, let me cite two examples.

Hilbert's second problem dealt with, among other things, the consistency of the axioms of number theory. Let us take the following as axioms for number theory (these are known as the Peano axioms):

- (a) 0 is a number
- (b) If n is number, then the successor of n (written $n + 1$) is also a number.
- (c) Different numbers have different successors.
- (d) (The Principle of Mathematical Induction). If P is a property true for 0, and if whenever it is true for n then it is true for $n + 1$, then P is true for all natural numbers $0, 1, \dots$

Hilbert believed that the Peano axioms were consistent, i.e. one could not prove, for example, $0 = 1$ from the axioms, and that the consistency could be proved from the axioms themselves. This led to his introduction of what is now known as the Hilbert programme. The programme was directed at the solution of the consistency problem, and it occupied him and a number of his very able assistants for almost two decades. The general feeling at the time was that a proof of the consistency of number theory (something which everyone believed) was only a matter of time. It therefore came as a shock in 1931 when the 25-year-old mathematician Kurt Gödel (1906 - 1978) destroyed all hope for the programme by showing that if number theory was consistent, then the consistency could not be proved within the system. In other words, one would have to go beyond the Peano axioms in order to prove their consistency, if indeed they were. The implications of this result were far reaching and profound. It is not an exaggeration to say that Gödel's proof offered a completely new way of looking at mathematics. His work also led to the introduction of a new mathematical field: mathematical logic. In the words of the great mathematician John von Neumann (1903 - 1957), Gödel's work was 'singular in the history of logic'.

The second example is the famous Riemann Hypothesis, number eight on the list of Hilbert problems. Riemann had, in the course of his study of the distribution of prime numbers, introduced the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s,$$

where $s = x + iy$ is a complex number. To those of you who have some acquaintance with infinite series, you would see that $\zeta(s) \neq 0$ for s a real number. Now the function as defined here clearly does not give a value for every s . For example, when $s = -1$, or -2 etc. the right hand side of the equation yields ∞ . But there is in the study of complex function theory the notion of 'analytic continuation', wherein the function $\zeta(s)$ can be

'continued analytically' to become a 'meromorphic function' defined on the whole complex plane. This new function coincides with the original one on the right-hand side of the complex plane, i.e. on all complex numbers $s = x + iy$ such that $x > 0$, and gives the value 0 whenever $s = -2, -4, \dots, -2n, \dots$. Let us denote this new function again by $\zeta(s)$. Riemann conjectured that except for $s = -2n$, the only time when $\zeta(s) = 0$ is when $s = \frac{1}{2} + iy$, the so-called 'critical line'. Hilbert had called this problem 'the most important problem in mathematics', and it remains unsolved today. (An analogue of the Riemann Hypothesis in algebraic geometry has been solved by the young Belgium mathematician Pierre Deligne (1946 -) a few years back.)

This brings us to the work of mathematicians of our time. By and large, we are still guided by the Hilbert problems, their ramifications and their subsequent developments. While a number of Hilbert problems have been solved, their solutions lead to still more penetrating and difficult questions. But of course not everyone works on the Hilbert problems. A rough estimate indicates that there are more than 100,000 mathematicians throughout the world, not all of them engaged in research. But the authoritative *Mathematical Reviews*, published by the American Mathematical Society, reviews about 1500 research papers each month, indicating that there is a great amount of mathematical research being produced daily by mathematicians from all countries. Not all of the work done are good of course. One would guess that perhaps many of the papers published in mathematical journals would be forgotten soon after publication. Collectively there is nevertheless an indication of the direction in which mathematics is going. The famous French mathematician Jean Dieudonné (1906 -) has recently published a paper entitled 'Current trends in pure mathematics' (*Advances in Mathematics*, volume 27, 1978), giving an overview of mathematics of today. About 13 mathematical disciplines were listed as belonging to the main-

stream of mathematics, and almost all the leading mathematicians of our time work in one or more of these areas.

While a hundred or more years ago it was not too difficult to explain to a layman a certain mathematical problem (for example a problem in elementary number theory), it is almost impossible today to enlighten the uninitiated on the importance of a mathematical problem without going through a few grueling hours of definitions and motivations. This is true especially in the case of problems in the so-called mainstream of mathematics. The subject has advanced so fast and become ever more technical that a beginning student in mathematics (say first year in the university) would need at least five years of intensive study before he can embark on a research project. In some areas of mathematics it takes an even longer time.

But there are of course always exceptional students who can finish in half the time or less. In the history of mathematics we have examples like Gauss, Henri Poincaré (1854 - 1912), von Neumann ("The fastest mind I have ever met," according to a physicist) etc. It is on the other hand true that quick minds do not necessarily make good mathematicians, and a great mathematician need not have a quick mind. Hilbert, for example, was known not to be particularly quick, yet his contributions certainly exceeded those of von Neumann in depth, if not in scope.

After spending seven or eight years (or longer) learning and doing mathematics, a student is awarded a Ph.D and is ready to pursue his own independent work. Most of the time, however, his experience in research will be rather limited but this handicap is compensated by the abundance of his youthful energy and enthusiasm in further improving himself through discussions and learning. An able mathematician today can find many op-

opportunities of learning from the experts by going, if chances permit, to a mathematical centre in Europe or America. There are establishments which have visitors from all over the world concentrating on research, for example the Institute for Advanced Study in Princeton, U.S.A.

To students who aspire to be mathematicians, I would suggest that hard work is the most important pre-requisite. Gauss, certainly one of the greatest in history, firmly believed that genius was industry. A number of famous contemporary mathematicians have expressed the view that even with only an above average intelligence, one could go quite far by working hard. This requires dedication to the profession of mathematics. In current economic context, there is no sense in producing a mathematician who after the Ph.D. decides to become mathematically stagnant, unless of course he takes up a job in industry or administration which requires a completely different kind of work. Currently there is a surplus of Ph.D.'s in mathematics throughout the developed countries, and in some developing countries as well. It therefore seems necessary to carefully monitor the situation in order that the bright and the dedicated be given a chance.

To the most ambitious there is the Fields Medal, awarded by the International Mathematical Union once every four years at the opening ceremony of the International Congress of Mathematicians. The Medal is popularly regarded as an equivalent of the Nobel Prize in mathematics (the Nobel Foundation, of course, does not award prizes in mathematics) and is given only to mathematicians below the age of forty. But at a lower level, a person should find great satisfaction in his work if he has done something really good. After a period of continued frustration, the final conquest of the problem attempted usually brings joy and pride that are difficult to describe. But of course most of the time even a modicum of genuine success is hard to come by.

In Singapore, whose mathematicians in the two universities have produced competent research work during the past 30 years or so, I think that we have come a long way. But much more work remains to be done. Perhaps one day we will be able to produce a mathematician of top calibre, something which Hong Kong has eminently achieved. This requires time and effort, but is not an impossible task. In fact it is hoped that some of you would take up the challenge. The going may be tough, but then nothing that is good is free, or easy.

CORRIGENDA: "WHAT IS A STAR?" BY K.K. SEN

The following are corrections to typographical errors in the article "What is a star?"

by K.K. Sen (this Medley, Vol. 6, No. 2, August 1978, pp. 62 - 68):

p. 63, line 7 from bottom:

$$\dots P_c \circlearrowleft = 2\bar{\rho} \circlearrowleft GM \circlearrowleft / R \circlearrowleft \dots;$$

p. 65, line 5 from bottom:

...Conduction is too slow a process...

p. 68, line 3 from top:

... transport equations (23) or