PROBLEMS AND SOLUTIONS

A book-voucher prize will be awarded to the best solution of a starred problem. Only solutions from Junior members and received before 1 December 1978 will be considered for the prizes. If equally good solutions are received, the prize or prizes will be awarded to the solution or solutions sent with the earliest postmark. In the case of identical postmarks, the winning solution will be decided by ballot.

Members are reminded that although no prizes are awarded to the contribution of problems, interesting problems at secondary school or university level are most welcome. Whenever possible, please submit a problem together with its solution. Problems or solutions should be sent to Dr. K. N. Cheng, Department of Mathematics, University of Singapore, Singapore 10.

*P6/78. Let T, T' be two linear transformations of the three-dimensional Euclidean space V into itself. Let A, A' be the matrices of T, T' respectively with respect to some fixed rectangular axes Oxyz with origin O. Further, let A' be the transpose of A. Prove that if P, Q, R are points of V such that $T : P \to Q$, $T' : P \to R$, then the line OP is perpendicular to the line QR.

(Via Ho Soo Thong)

*P7/78. Given ΔABC. Let L, M be points on AB, BC respectively such that $AL : LB = BM : MC = k$, where k is some finite non-zero real number. If P, Q are points on AB, BC respectively such that PQ intersects LM at T with $PT : TQ = k$, show that $AP : PB = BQ : QC = LT : TM$.

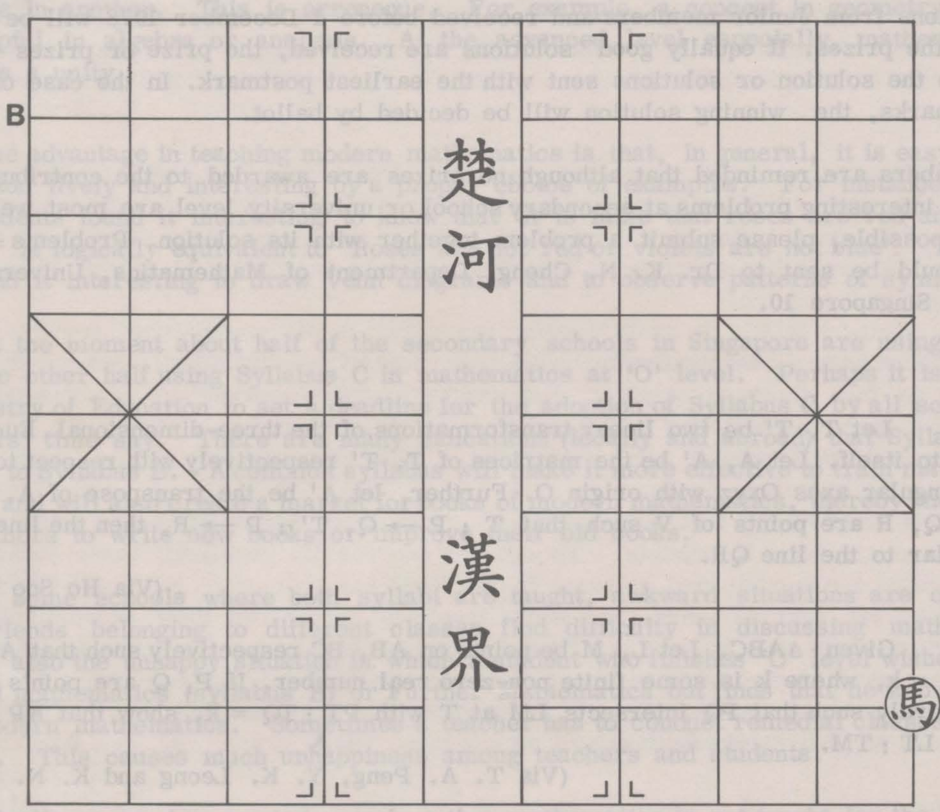
(Via T. A. Peng, Y. K. Leong and K. N. Cheng)

*P8/78. A sphere of mass m_1 moves with uniform velocity V on a smooth horizontal floor and impinges upon a stationary sphere of mass m_2, which then moves on the floor until it hits a fixed vertical plane barrier. In order to allow as many impacts as possible, the barrier is shifted parallel to itself through some distance away from the rebounding sphere after each impact and kept fixed before the next impact. Assuming that all collisions are direct and perfectly elastic, find the total number of collisions between the two spheres.

(Via Y. K. Leong)

Solutions to P1 - P5/78 :

We refer problems P1, P2/78 to the following chinese chessboard :



*P1/78. What is the minimum number of steps required to move a Horse from point A to point B?

Solution of Goh Koon Shim :

Let us regard the chessboard as divided into 9 x 8 = 72 unit squares (refer Fig. 1) and call the vertices of the squares the points on the board.

We say that two points on the board are adjacent to each other if they are one unit apart. By a unit step we shall mean a jump from one point to another which is adjacent to it. Then it takes at least 9 unit steps horizontally and 6 unit steps vertically to reach B from A (see Fig. 2), i.e. a total of at least 15 unit steps. Now in one move a Horse makes exactly 3 unit steps, namely 2 unit steps horizontally (or vertically) and 1 unit step vertically (or horizontally). Thus we need at least 15/3 = 5 steps to move a Horse from A to B. Since there exists at least one way of moving a Horse from A to B in five steps (see for example Fig. 2), the minimum number of steps required is five.

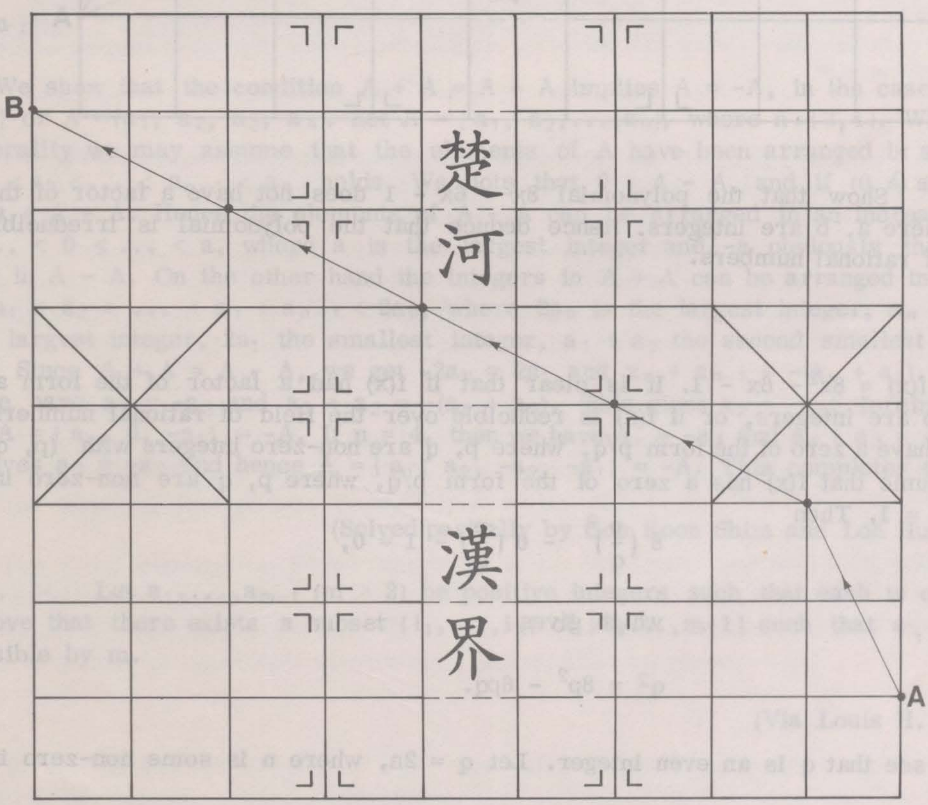
(Also solved by Loh Hung Leong and Tay Yong Chiang)

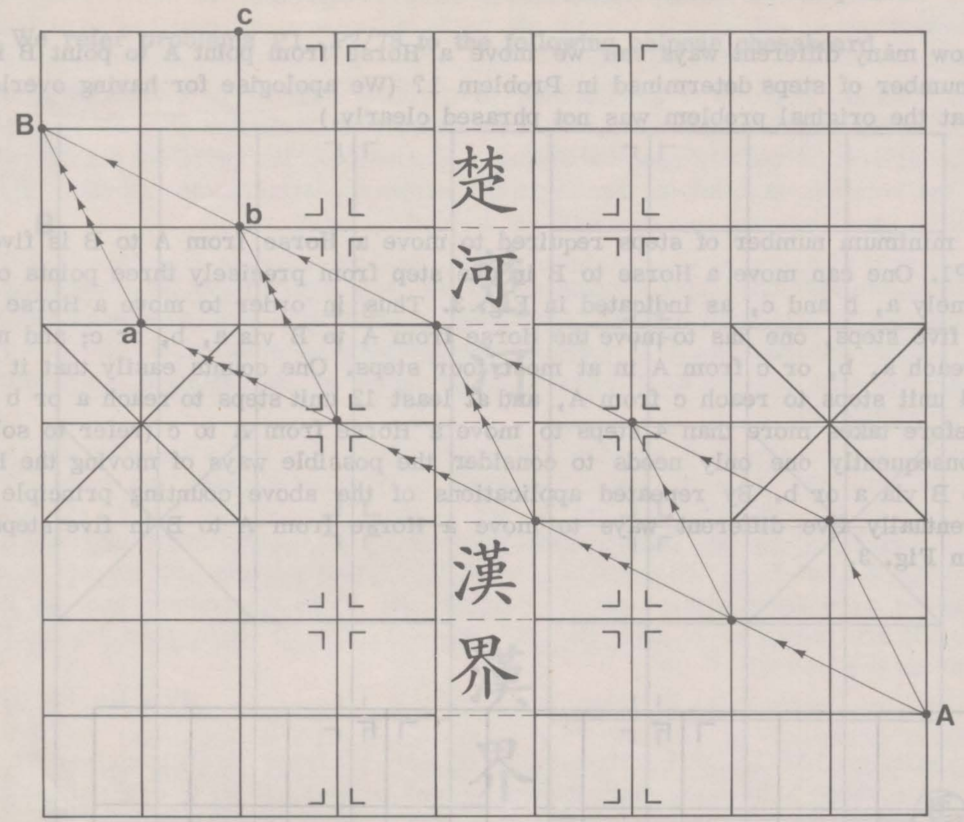
*P2/78. This problem should read :

In how many different ways can we move a Horse from point A to point B in the minimum number of steps determined in Problem 1? (We apologise for having overlooked the fact that the original problem was not phrased clearly.)

Solution :

The minimum number of steps required to move a Horse from A to B is five, as shown in P1. One can move a Horse to B in one step from precisely three points on the board, namely a, b and c, as indicated in Fig. 3. Thus in order to move a Horse from A to B in five steps, one has to move the Horse from A to B via a, b, or c; and moreover, to reach a, b, or c from A in at most four steps. One counts easily that it takes at least 14 unit steps to reach c from A, and at least 12 unit steps to reach a or b from A. It therefore takes more than 4 steps to move a Horse from A to c (refer to solution of P1). Consequently one only needs to consider the possible ways of moving the Horse from A to B via a or b. By repeated applications of the above counting principle, one obtains eventually five different ways to move a Horse from A to B in five steps, as indicated in Fig. 3.





*P3/78. Show that the polynomial $8x^2 - 6x - 1$ does not have a factor of the form $ax + b$, where a, b are integers. Hence deduce that the polynomial is irreducible over the field of rational numbers.

Solution :

Set $f(x) = 8x^2 - 6x - 1$. It is clear that if f(x) had a factor of the form $ax + b$, where a, b are integers, or if f(x) is reducible over the field of rational numbers, then f(x) would have a zero of the form p/q, where p, q are non-zero integers with $(p, q) = 1$. Let us assume that f(x) has a zero of the form p/q, where p, q are non-zero integers with $(p, q) = 1$. Then

$$8\left(\frac{p}{q}\right)^2 - 6\left(\frac{p}{q}\right) - 1 = 0,$$

which gives

$$q^2 = 8p^2 - 6pq.$$

We see that q is an even integer. Let $q = 2n$, where n is some non-zero integer. Then

$$(2n)^2 = 8p^2 - 12pn,$$

so that

$$n(n - 3p) = 2p^2 \ldots\ldots\ldots\ldots (1)$$

We now look at equation (1). Since (p, q) = 1 and q = 2n, we must have $(p^2, n) = 1$. This forces $n \in \{\pm 1, \pm 2\}$. If $n = \pm 1$, equation (1) becomes $1 = 2p^2 \pm 3p = p(2p \pm 3)$, if $n = \pm 2$, we get $2 = p(p \pm 3)$. One verifies easily that both equations cannot be satisfied by an integer p. It follows from this contradiction that f(x) has no factor of the form ax + b, where a, b are integers. This implies also that f(x) is irreducible over the field of rational numbers.

*P4/78. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of n distinct integers. We define the sets -A, A + A and A - A as follows :

$$-A = \{-a_i \mid a_i \in A\}$$

$$A + A = \{a_i + a_j \mid a_i, a_j \in A\}$$

$$A - A = \{a_i - a_j \mid a_i, a_j \in A\}$$

For example of A = {-5, -3, -1, 3, 5}, then A + A = {-10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10} = A - A. In this case we see that $A \neq -A$. Prove that there does not exist A satisfying

$$A + A = A - A \text{ and } A \neq -A,$$

with $A = \{a_1, a_2, a_3\}$ or $A = \{a_1, a_2, a_3, a_4\}$.

(Via H. P. Yap)

Solution :

We show that the condition A + A = A - A implies A = -A, in the case $A = \{a_1, a_2, a_3\}$ or $A = \{a_1, a_2, a_3, a_4\}$. Set $A = \{a_1, a_2, \ldots, a_n\}$, where $n \in \{3, 4\}$. Without loss of generality we may assume that the elements of A have been arranged in such a way that $a_1 < a_2 < \ldots < a_{n-1} < a_n$ holds. We note that $0 \in A - A$, and if $(0 \neq)\, a \in A - A$, then $-a \in A - A$. Hence the elements of A - A can be arranged in an increasing order $-a < \ldots < 0 < \ldots < a$, where a is the largest integer and -a obviously the smallest integer in A - A. On the other hand the integers in A + A can be arranged in the order $2a_1 < a_1 + a_2 < \ldots < a_1 + a_{n-1} < 2a_n$, where $2a_n$ is the largest integer, $a_n + a_{n-1}$ the second largest integer, $2a_1$ the smallest integer, $a_1 + a_2$ the second smallest integer of A + A. Since A + A = A - A, we get $-2a_n = 2a_1$ and $a_n + a_{n-1} = -(a_1 + a_2)$. If n = 3, then we have $a_1 = -a_3$ and $a_3 + a_2 = -(a_1 + a_3)$. This gives $a_2 = -a_3$, forcing $a_2 = 0$. Hence $A = \{a_1, 0, -a_1\} = -A$. If n = 4, then we have $a_1 = -a_4$ and $a_4 + a_3 = -(a_1 + a_2)$. This gives $a_3 = -a_2$ and hence $A = \{a_1, a_2, -a_2, -a_1\} = -A$. This completes the proof.

(Solved partially by Goh Koon Shim and Loh Hung Leong)

*P5/78. Let $a_1, \ldots, a_{m-1}$ $(m > 2)$ be positive integers such that each is coprime to m. Prove that there exists a subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, m-1\}$ such that $a_{i_1} \ldots a_{i_k} - 1$ is divisible by m.

(Via Louis H. Y. Chen)

Solution :

We are given (m - 1) positive integers $a_1,\ldots,a_{m-1}$ (m > 2) such that each is coprime to m. Let us define (m - 1) integers $b_1,\ldots,b_{m-1}$ by $b_i = \prod_{k=1}^{i} a_k$, for each $i \in \{1,\ldots,m-1\}$. Then each b_i is a positive integer prime to m. For each $i \in \{1,\ldots,m-1\}$, let $b_i \equiv r_i$ (mod m), where $0 \leqslant r_i \leqslant m - 1$. Since $(b_i, m) = 1$, we have $r_i \geqslant 1$. There are therefore precisely (m - 1) integers $r_1,\ldots,r_{m-1}$, where each of them lies in $\{1,\ldots,m - 1.\}$ Thus one of the following cases must occur :

(i) there exists some $k \in \{1,\ldots,m - 1\}$ such that $r_k = 1$;

(ii) there exist distinct integers k and l in $\{1,\ldots,m - 1\}$ such that $r_k = r_l$.

If (i) holds, then $b_k = \prod_{j=1}^{k} a_j \equiv 1$ (mod m), and we are done. If (ii) occurs, then $b_k \equiv b_l$ (mod m). Assume without loss of generality that $l > k$. Then we have $a_1 a_2 \ldots a_k = a_1 a_2 \ldots a_k a_{k+1} \ldots a_l$ (m). Since $(a_i, m) = 1$ for each $i \in \{1,\ldots,m - 1\}$, it follows that $a_{k+1} \ldots a_l \equiv 1$ (mod m). This completes the proof.

Remark In connection with this problem we would like to mention the following theorem due to Euler :

Let m be a positive integer larger than 1. If a is a positive integer such that (a, m) = 1, then $a^{\phi(m)} \equiv 1 (m)$*, where* $\phi(m)$ *denotes Euler's* ϕ*-function of m, which is the number of positive integers which are less than m and prime to m.*

A book-voucher prize of $5 has been awarded to Goh Koon Shim for the solution of of Problem 1.