INTER-SCHOOL MATHEMATICAL COMPETITION 1978

The problems of the Inter-school Mathematical Competition 1978 are reproduced below. The Competition Subcommittee expresses its regret that the original version of Question 1, Part B, was incorrect due to an oversight. The correct statement of this question is given below.

Correct answers in Part A are marked with asterisks. Outlines of solutions to problems in Part B are included.

Part A

Saturday, 1 July 1978 — 2.00 p.m. - 3.00 p.m.

1. Find the value of

$$\log_2(\log_2 \underbrace{\sqrt{\sqrt{\cdots\sqrt{2}}}}_{k \text{ times}}) .$$

(a) 1 ; (b) 2^k ; (c)* $-k$; (d) $1/k$; (e) 0

2. In how many ways can 3 distinct numbers be selected from the numbers $1,2,\ldots,3n$ such that their sum is divisible by 3?

(a) $\left[\binom{n}{3}\right]^3$; (b) $3\binom{n}{3}$; (c) $\binom{n}{3}+n^3$; (d)* $3\binom{n}{3}+n^3$; (e) none of the above.

3. If a,b and c are different digits (from 1 to 9 inclusive), find the largest possible value of $(a+b+c)/(abc)$.

(a) 3 ; (b) 2 ; (c)* 1 ; (d) $\frac{3}{4}$; (e) none of the above.

4. The driver of a bus enters a parking lot consisting of 12 places in a row, and observes that only 8 places are occupied. What is the probability that he will be able to park his bus if it requires 4 places?

(a) $\frac{1}{3}$; (b) $\frac{1}{9}$; (c) $\frac{1}{4}$; (d) $4/495$; (e)* $1/55$

5. Let S be a set. A relation on S is a subset R of the cartesian product S x S. R is <u>reflexive</u> if $(a, a) \in R$ for all $a \in S$. If S has n elements, find the number of reflexive relations R.

(a) n^2 ; (b) 2^{n^2} ; (c) $\frac{1}{2}n(n+1)$; (d)* 2^{n^2-n} ; (e) none of the above.

6. How many triangles of different shapes and of perimeter the length of seven identical matchsticks can be formed using exactly seven matchsticks?

(a) 0 ; (b) 1 ; (c)* 2 ; (d) 3 ; (e) none of the above.

7. In a tug-of-war, a man of weight W kg pulls the rope horizontally at a height a cm above the ground. If b cm is the horizontal projection of the line joining his heels to his centre of gravity, what is the pull exerted on the rope?

(a)* Wb/a kg ; (b) Wa/b kg ; (c) Wb^2/a kg ; (d) Wa^2/b kg ;
(e) none of the above.

8. Two spheres (one a hollow shell and the other a homogeneous solid) of the same radius roll down an incline together, starting simultaneously from rest at the top. Then

(a) the hollow sphere will reach the bottom first
(b)* the homogeneous sphere will reach the bottom first
(c) they will reach the bottom at the same time
(d) the time of descent of each sphere depends on its mass
(e) the time of descent of each sphere depends on the coefficient of friction between the sphere and incline.

9. Find the maximum number of points of intersection of 7 circles drawn on the surface of a sphere.

(a) 14 ; (b) 21 ; (c)* 42 ; (d) 49 ; (e) 2^7

10. If PQ is a diameter of the base of a right circular cylinder and R is a point on the cylinder vertically above Q with QR = π cm, PQ = 2 cm, find the length of the shortest path on the surface of the cylinder from P to R.

(a) $(\pi + 2)$ cm ; (b) $\sqrt{\pi^2+4}$ cm ; (c)* $\pi\sqrt{2}$ cm ; (d) 2π cm ; (e) $\pi\sqrt{3}$ cm

Part B

Saturday, 1 July 1978 3.00 p.m. - 5.00 p.m.

1. Let f(x) be a polynomial of degree $n(\geqslant 1)$. Denote by $f^{(i)}(x)$ the ith derivative of f(x). If a and b are real numbers such that $a < b$, $f(a) < 0$, $f(b) > 0$ and

$$f'(a) \geqslant 0,\ (-1)f''(a) \geqslant 0, \ldots,\ (-1)^{n-1} f^{(n)}(a) \geqslant 0,$$

$$f'(b) \geqslant 0,\ f''(b) \geqslant 0, \ldots\ldots,\ f^{(n)}(b) \geqslant 0,$$

prove that f(x) = 0 has all its real roots between a and b.

2. Let a and b be real numbers such that $a^2 + b^2 = 1$. Are there any values of a and b for which cos a - sin b is negative?

3. ABC is a triangle of unit area. Points P, Q, R are taken on the sides BC, CA, AB respectively such that

$$BP : PC = CQ : QA = AR : RB .$$

Show that the area of the triangle PQR is not less than $\frac{1}{4}$.

4. Let A be a set of n positive integers. Prove that there is a nonempty subset B of A such that the sum of the elements of B is divisible by n.

5. If x is any real number, $[x]$ denotes the largest integer m satisfying the equality $x = m + r$, where $0 \leqslant r < 1$.

Prove that if a and b are real numbers, then

(i) $[a + b] \geqslant [a] + [b]$,

(ii) $\left[a + \frac{1}{a}\right] \geqslant 2$ if $a > 0$,

(iii) $\left[a + \frac{1}{a}\right] \leqslant -3$ if $a < -1$,

(iv) $[2a + 2b] \geqslant [a] + [b] + [a + b]$.

Solutions to Part B

1. Write $f(x) = a_o + a_1(x - a) + \dots + a_n (x - a)^n$.

Then $a_o = f(x)$ and $a_i = f^{(i)}(a)/i!$ for $i \geqslant 1$, so that

$$(*) \quad f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + + \frac{f^{(n)}(a)}{n!}(x - a)^n .$$

Similarly

$$(**) \quad f(x) = f(b) + \frac{f'(b)}{1!}(x - b) + \frac{f''(b)}{2!}(x - b)^2 + \dots + + \frac{f^{(n)}(b)}{n!}(x - b)^n .$$

$(*)$ shows that all real roots of $f(x) = 0$ are greater than a, and $(**)$ shows that all real roots are less than b.

2. $$\begin{aligned} \cos a - \cos b &= \cos a + \cos(\tfrac{1}{2}\pi + b) \\ &= 2\cos\tfrac{1}{2}(\tfrac{1}{2}\pi + a + b) \cdot \cos\tfrac{1}{2}(\tfrac{1}{2}\pi - (a - b)). \end{aligned}$$

$$\begin{aligned} \text{Now } |a \pm b| &= \sqrt{(a \pm b)^2} \\ &= \sqrt{a^2 + b^2 \pm 2ab} \\ &\leqslant \sqrt{2(a^2 + b^2)} = \sqrt{2} < \tfrac{1}{2}\pi, \end{aligned}$$

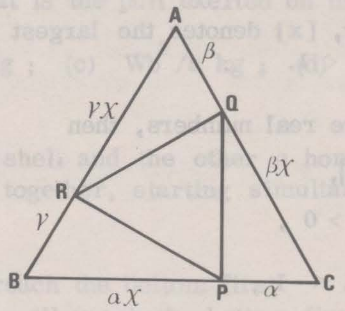
since $a^2 + b^2 \geqslant 2ab$. Thus

$$0 < \tfrac{1}{2}(\tfrac{1}{2}\pi + a + b) < \tfrac{1}{2}\pi ,$$

$$0 < \tfrac{1}{2}(\tfrac{1}{2}\pi - (a - b)) < \tfrac{1}{2}\pi .$$

Hence $\cos a - \sin b > 0$. There are no values of a and b for which $\cos a - \sin b$ is negative.

3.



Let $BP : PC = CQ : QA = AR : RB = x$, and let $PC = \alpha$, $QA = \beta$, $RB = \gamma$.

Then Δ = area of ΔABC

$= \frac{1}{2}\beta\gamma(1+x)^2 \sin A$

Area of $\Delta ARQ = \frac{1}{2}\beta\gamma x \sin A$

$= \dfrac{x}{(1+x)^2}\Delta$.

Similarly, area of ΔBPR = area of ΔCQP

$= \dfrac{x}{(1+x)^2}\Delta$.

Thus area of $\Delta PQR = \Delta - \dfrac{3x}{(1+x)^2}\Delta$.

Since $\dfrac{x}{(1+x)^2} \leqslant \frac{1}{4}$ (by calculus or otherwise), and $\Delta = 1$, area of $\Delta PQR \geqslant \frac{1}{4}$.

4. Let $A = \{a_1, \ldots\ldots, a_n\}$. Consider the remainders $r_1, r_2, \ldots\ldots, r_n$ when the respective numbers $a_1, a_1 + a_2, \ldots\ldots, a_1 + a_2 + \ldots\ldots + a_n$ are divided by n. If $r_1, \ldots, r_n$ are distinct, then $r_i = 0$ for some $1 \leqslant i \leqslant n$. If $r_j = r_k$ for some $1 \leqslant j < k \leqslant n$, then $a_{j+1} + \ldots\ldots + a_k$ is divisible by n.

5. (i) Write $a = m + r$, $b = n + s$, where m, n are integers and $0 \leqslant r, s < 1$. Then $a + b = m + n + \ell + t$, where $r + s = \ell + t$ with ℓ a non-negative integer, $0 \leqslant t < 1$. Hence $[a + b] = m + n + \ell \geqslant m + n = [a] + [b]$.

(ii) Let $a = m + r$, $0 \leqslant r < 1$, m non-negative integer.

Then $a + \frac{1}{a} = \frac{(m + r)^2 + 1}{m + r}$

$$= 2 + \frac{(m + r - 1)^2}{m + r} \geqslant 2$$

(iii) If $a < -1$, then $a = -m + r$ where $m \geqslant 2$ is an integer and $0 \leqslant r < 1$.

Now $a + \frac{1}{a} = -m + s$,

where $s = r + \frac{1}{r - m}$.

If $m = 2$, $s < 0$.

If $m \geqslant 3$, $s = 1 - \frac{(m - r)(1 - r) + 1}{m - r} < 1$.

Hence $\left[a + \frac{1}{a}\right] \leqslant -3$.

(iv) Write $a = m + r$, $b = n + s$; m, n integers and $0 \leqslant r, s < 1$. Then $[a + b] = m + n$ or $m + n + 1$ according as $r + s < 1$ or $r + s \geqslant 1$. Since $[2a + 2b] \geqslant 2m + 2n + 2$ if $r + s \geqslant 1$, the result follows.