TEACHING NOTES

TWO CUBE ROOT FORMULAE FOR USERS OF SIMPLE ELECTRONIC CALCULATORS

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Many a time pupils have come up to me asking whether it would be possible for them to extract cube roots of numbers on simple electronic calculators (i.e. those without $\sqrt{\ }$ and Y^x keys). I have heard a salesman telling a prospective buyer of simple electronic calculators that square roots & cube roots cannot be performed on these calculators.

The following two formulae for cube roots do not seem to be well-known among pupils.

(1) If $\sqrt[3]{x} \simeq a$, then $\frac{1}{3}\left(\frac{x}{a^2} + 2a\right)$ is a better approximation.

(2) $\sqrt[3]{1 + x} \simeq \frac{6 + 7x}{6 + 5x}$ where $0 < x < 1$.

If $\sqrt[3]{x} \simeq a$, then $\sqrt[3]{x} = a + h$ where h is the small error. Thus

$$x = (a + h)^3 ,$$

and

$$x \simeq a^3 + 3a^2h ,$$

$$h \simeq \frac{x - a^3}{3a^2} .$$

Hence

$$\sqrt[3]{x} \simeq \frac{1}{3}\left(\frac{x}{a^2} + 2a\right) .$$

The proof of (2) is not difficult. It is within the reach of pupils who have done the binomial expansion, and is left as an exercise.

Suppose we want to find $\sqrt[3]{68}$. We all know that

$\sqrt[3]{68} \simeq 4$ so $a = 4$. Thus $\frac{1}{3}\left(\frac{x}{a^2} + 2a\right) = \frac{1}{3}\left(\frac{68}{4^2} + 8\right) = 4.083$

is a better approximation which is correct to 3 significant figures. If we repeat the process by taking $a = 4.083$,

then $\frac{1}{3}\left(\frac{68}{4.083^2} + 2(4.083)\right) = 4.081655$ which is correct to 7 significant figures.

Let us find the same cube root by using formula (2):

$$\sqrt[3]{68} = \sqrt[3]{64 + 4} = \sqrt[3]{64\left(1 + \frac{1}{16}\right)}$$

$$= 4\sqrt[3]{1 + \frac{1}{16}}$$

$$= 4\left(\frac{6 + \frac{7}{16}}{6 + \frac{5}{16}}\right) = 4\left(\frac{103}{101}\right)$$

$= 4.079$ which is correct to 2 significant figures.

So formula (1) gives a better approximation than formula (2) generally, provided we start with a good approximation. Formula (2) always gives an error in defect less than $\frac{2x^3}{81}$.

My pupils enjoy the proofs and application of these formulae. I hope this brief note will help others to enjoy mathematics & its applications.