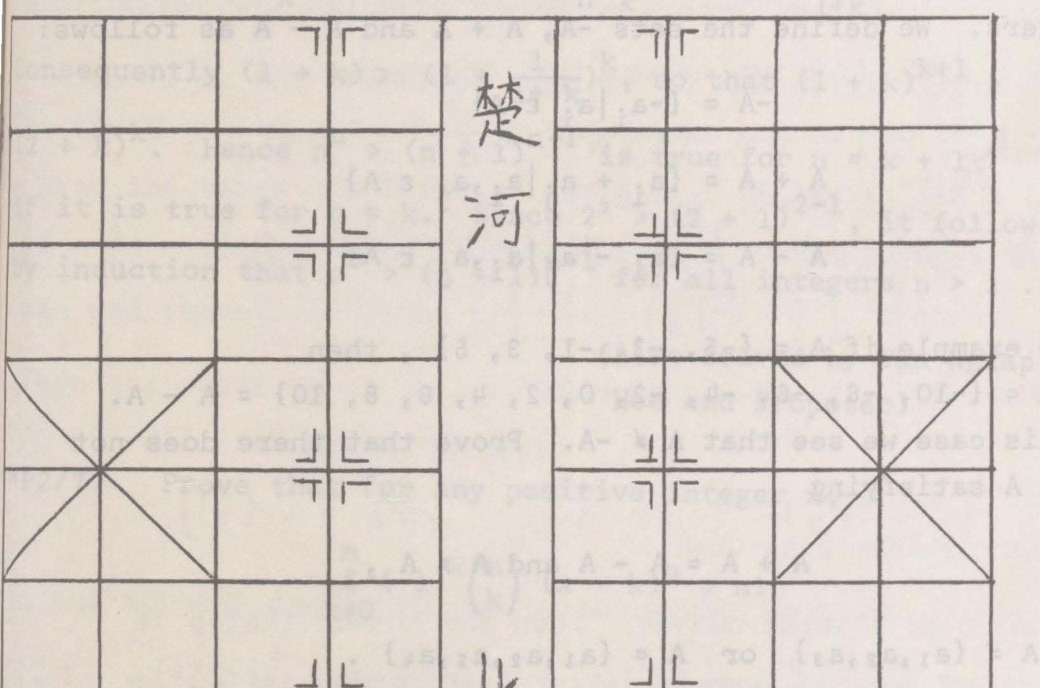
PROBLEMS AND SOLUTIONS

A book-voucher prize will be awarded to the best solution of a starred problem. Only solutions from Junior members and received before 1 August 1978 will be considered for the prizes. If equally good solutions are received, the prize or prizes will be awarded to the solution or solutions sent with the earliest postmark. In the case of identical postmarks, the winning solution will be decided by ballot.

Problems or solutions should be sent to Dr. K. N. Cheng, Department of Mathematics, University of Singapore, Singapore 10. Whenever possible, please submit a problem together with its solution.

We refer problems P1, P2/78 to the following chinese chessboard :



*P1/78. What is the minimum number of steps required to move a Horse from point A to point B?

*P2/78. How many different ways can we move a Horse from point A to point B?

(For both problems you may assume that the Horse can be moved without any blocking.)

(via H. P. Yap)

*P3/78. Show that the polynomial $8x^2 - 6x - 1$ does not have a factor of the form $ax + b$, where a, b are integers. Hence deduce that the polynomial is irreducible over the field of rational numbers.

*P4/78. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of n distinct integers. We define the sets $-A$, $A + A$ and $A - A$ as follows:

$$-A = \{-a_i \mid a_i \in A\}$$

$$A + A = \{a_i + a_j \mid a_i, a_j \in A\}$$

$$A - A = \{a_i - a_j \mid a_i, a_j \in A\}$$

For example if $A = \{-5, -3, -1, 3, 5\}$, then $A + A = \{-10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10\} = A - A$. In this case we see that $A \neq -A$. Prove that there does not exist A satisfying

$$A + A = A - A \text{ and } A \neq A,$$

with $A = \{a_1, a_2, a_3\}$ or $A = \{a_1, a_2, a_3, a_4\}$.

(via H. P. Yap)

*P5/78. Let $a_1, \ldots, a_{m-1}$ $(m > 2)$ be positive integers such that each is coprime to m. Prove that there exists a subset $\{i_1,\ldots,i_k\}$ of $\{1,\ldots,m-1\}$ such that $a_{i_1}\ldots a_{i_k} - 1$

is divisible by m.

(via Louis H. Y. Chen)

Solutions to P1 - P4/77 :

*P1/77 Prove, preferably by induction, that for all integers $n > 1$, $n^n > (n + 1)^{n-1}$.

(via Chan Sing Chun)

Solution by Tay Yong Chiang :

Suppose for some integer $k > 1$, $k^k > (k + 1)^{k-1}$. Then

$(1 + k) > (1 + \frac{1}{k})^k$. Clearly $(1 + \frac{1}{k})^k > (1 + \frac{1}{1+k})^k$.

Consequently $(1 + k) > (1 + \frac{1}{1+k})^k$, so that $(1 + k)^{k+1} >$

$(2 + k)^k$. Hence $n^n > (n + 1)^{n-1}$ is true for $n = k + 1$,

if it is true for $n = k$. Since $2^2 > (2 + 1)^{2-1}$, it follows

by induction that $n^n > (n + 1)^{n-1}$ for all integers $n > 1$.

(Also solved by Tan Ngiap Hoe and Proposer)

*P2/77 Prove that for any positive integer n,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x - k)^n = n!$$

(via Chan Sing Chun)

Solution by Y. K. Leong :

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x - k)^n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} x^{n-r} \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^r$$

Here $\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^r = \frac{d^r}{dt^r}\left[(1 - e^t)^n\right]\Big|_{t=0} =$

$$= (-1)^n \frac{d^r}{dt^r}\left\{t^n(1 + \frac{t}{2!} + \frac{t^2}{3!} + \ldots)^n\right\}\Big|_{t=0} =$$

$$= \begin{cases} 0 & \text{if } r = 0,1,\ldots,n-1 \ ; \\ (-1)^n n! & \text{if } r = n \ . \end{cases}$$

The result follows.

*P3/77 Two positive integers are said to be relatively prime if their only common factor is 1. What is the probability that two numbers selected at random are relatively prime? (You may assume that the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ converges to $\frac{\pi^2}{6}$ and that

$$\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right)\left[\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right) \ldots\right] = 1 .)$$

(via L. Y. Lam)

Solution by Chan Sing Chun

Let p be a prime number. Then the chance that p is a factor of the two numbers selected at random is $(\frac{1}{p})^2$. Thus the chance that p is not a factor is $(1 - \frac{1}{p^2})$. Hence the chance that two positive integers selected at random are relatively prime is $\prod_{p \text{ prime}}(1 - \frac{1}{p^2})$, which is equal to $\frac{6}{\pi^2}$

by the given identity.

P4/77 Five sailors plan to divide a pile of coconuts among themselves in the morning. During the night one of them wakes up and decides to take his share. After throwing a coconut to a monkey to make the division come out even, he takes one fifth of the pile and goes back to sleep. The other four sailors do likewise, one after the other, each throwing a coconut to the monkey and taking one fifth of the remaining pile. In the morning the five sailors throw a coconut to the monkey and divide the remaining coconuts into five equal piles. What is the smallest number of coconuts that could have been in the original pile?

(via K. Y. Woo)

Solution by Hooi Lai Ngoh :

Let x be the number of coconuts in the original pile. The first sailor who wakes up throws one coconut to the monkey and takes one fifth of the pile. He leaves $\frac{4}{5}(x-1) = (\frac{4}{5}x - \frac{4}{5})$ coconuts. The second sailor who wakes up does the same and therefore leaves $\left[(\frac{4}{5})^2 x - (\frac{4}{5})^2 - \frac{4}{5}\right]$ coconuts. After the other three sailors have done likewise, the number of coconuts left would be

$$\left[(\tfrac{4}{5})^5 x - (\tfrac{4}{5})^5 - (\tfrac{4}{5})^4 - (\tfrac{4}{5})^3 - (\tfrac{4}{5})^2 - \tfrac{4}{5}\right]$$

In the morning one is thrown to the monkey leaving

$$\left[(\tfrac{4}{5})^5 x - \left\{(\tfrac{4}{5})^5 + (\tfrac{4}{5})^4 + (\tfrac{4}{5})^3 + (\tfrac{4}{5})^2 + \tfrac{4}{5} + 1\right\}\right] \text{ coconuts.}$$

This is divided into 5 equal piles. Let there be n coconuts in each of these piles.

Then

$$\left(\frac{4}{5}\right)^5 x - \left(\frac{4}{5}\right)^5 + \left(\frac{4}{5}\right)^4 + \left(\frac{4}{5}\right)^3 + \left(\frac{4}{5}\right)^2 + \frac{4}{5} + 1 = 5n ,$$

which gives

$$\left(\frac{4}{5}\right)^5 x - 5\left[1 - \left(\frac{4}{5}\right)^6\right] = 5n.$$

Multiplying both sides by $\left(\frac{5}{4}\right)^5$ and rearranging we get

$$x = \frac{5^6(n + 1)}{4^5} - 4.$$

In order that x has the smallest integral value, $\frac{(n + 1)}{4^5}$ must be equal to 1. This gives $n + 1 = 4^5$ and $x = 15621$.

Alternate solution by Chan Sing Chun :

Let the smallest number of coconuts in the original pile be x.

A coconut was thrown to the monkey six times "to make the division come out even". So if there were 4 more coconuts in the pile the monkey would have none because the division would come out even each time a division was made. Hence $x + 4 = 5^6$ and $x = 15621$.