1977 INTER-SCHOOL MATHEMATICAL COMPETITION

The problems of the 1977 Inter-school Mathematical Competition are reproduced below. In Part 1, each correct answer is marked with an asterisk. Outlines of the solutions to the problems of Part 2 are included.

Part 1

Saturday, 16 April 1977 2.00p.m. - 3.00p.m.

1. The smallest sum of squares of two distinct positive primes is

(a) 5; (b) 8; (c)* 13; (d) 34;
(e) none of the above.

2. The greatest common divisor of

$$2^{2^2}, \quad 2^{2^{2^2}}, \quad 2^{2^{2^{2^2}}}, \quad \ldots$$ is

(a) 4 ; (b)* 16 ; (c) indeterminate ; (d) ∞ ;
(e) 8 .

3. The remainder obtained when 2^{30} is divided by 7 is

(a)* 0 ; (b) 1 ; (c) -1 ; (d) 6 ;
(e) none of the above.

4. If $n \geqslant 2$ is an integer, then

$$n(n^2 - 1)(n^2 + 1)$$

is always divisible by

(a) 4 ; (b) 8 ; (c) 9 ; (d)* 15 ; (e) 21 .

5. The number of extremum points (that is, maximum or minimum points) of the graph of

$p(x) = a_0x^{2n} + a_1x^{2n-1} + \ldots + a_{2n-1}x + a_{2n}$ is

(a) even ; (b) odd ; (c) n-1 ; (d) n ;
(e)* none of the above.

6. The value of $\int_0^{\pi} \sqrt{\sin x}\, dx$ is

(a) 2 ; (b) 0 ; (c)* $\leq \pi$; (d) ≤ 1 ;
(e) none of the above.

7. A pendulum clock is taken from the earth and placed on the moon. This clock on the moon will

(a) continue to keep correct time ;
(b) continue to keep correct time for 24 hours and then begin to lose time;
(c)* continue to keep correct time for 24 hours and then begin to gain time;
(d) gain time;
(e) lose time.

8. Let $a < x_0 < b$ and $x_n = \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right)$ for n = 1,2,3, ...

Then $a \leq x_n \leq b$ provided that

(a) a = 0 , b = 1 ; (b)* a = 1 , b = 2 ;
(c) a = 2 , b = 3 ; (d) a = 3 , b = 4 ;
(e) a = 4 , b = 5 .

9. A car travels from A to B with a uniform speed of 40km/hr and returns from B to A with a uniform speed of 30km/hr. What is the average speed of the car for the round trip? (Give your answer correct to 1 decimal place.)

(a) 34.1km/hr ; (b) 35km/hr ; (c) 35.1km/hr ;
(d)* 34.3km/hr ; (e) none of the above.

10. If $F(x) = \int_0^x xf(t)dt$, then $F'(x)$ is

(a) $xf(x)$; (b) $xf(t)$; (c) $tf(t)$;

(d)* $xf(x) + \int_0^x f(t)dt$; (e) none of the above .

11. If $\int_a^b f(x)dx = 3$ and $\int_c^d g(x)dx = 2$, then the value of $\int_a^b \left(\int_c^d f(x)g(u)du \right) dx$ is

(a) 5 ; (b) 3 ; (c) 2 ; (d) cannot be determined from the given data ; (e)* none of the above .

12. A balanced coin is tossed indefinitely. Find the probability that the 10^{th} head is followed by 5 consecutive heads.

(a) $(0.5)^{15}$; (b) $(0.5)^{10} + (0.5)^5$; (c)* $(0.5)^5$;
(d) $(0.5)^{10}$; (e) none of the above .

13. A ladder falls freely under gravity through a certain height with its length in a vertical position, and a monkey sits on the lowest rung of the ladder. Let T_0 be the time taken by the ladder to reach the ground if the monkey remains at the lowest rung, and T_1 the corresponding time taken if the monkey climbs up the ladder slowly. Which of the following is true ?

(a) $T_0 < T_1$; (b)* $T_0 > T_1$; (c) $T_0 = T_1$;
(d) $T_0 < 2T_1$; (e) $T_0 > 3T_1$.

14. Three sides CA, AB and BC of the triangle ABC are respectively 1, x, and $\sqrt{1 + xy + x^2}$ $(-2 < y < 2)$. The angle A

(a) depends on both x and y ;
(b) depends on x, but not on y ;
(c)* depends on y, but not on x ;
(d) does not depend on x or y ;
(e) cannot be determined by the given data .

15. If α and β are angles satisfying the equation

$$a \cos 4\theta + 3a \cos 2\theta + b = 0 ,$$

and if $\cos^2\alpha \neq \cos^2\beta$, then the value of $\cos^2\alpha + \cos^2\beta$ is

(a) $8a + 4b$; (b) $4a + b$; (c)* 1 ; (d) $\frac{4a + b}{20a}$;

(e) 3/2 .

16. Let $G_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right\}$,

$G_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$

be two groups under matrix multiplication. Then

(a)* G_1 and G_2 are isomorphic because both are cyclic of order 4 ;

(b) G_1 and G_2 are not isomorphic because $i = \sqrt{-1}$ cannot be equal to 1 or -1 ;

(c) G_1 and G_2 are not isomorphic because they have different elements ;

(d) G_1 and G_2 are isomorphic because both groups are commutative ;

(e) G_1 and G_2 are isomorphic because each of them contains four matrices.

17. The number of solutions of the matrix equation

$X^n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, where X is a 2×2 real matrix and

$n \geqslant 2$, is

(a) 1 ; (b) n ; (c)* greater than n ; (d) between 2 and n inclusive ; (e) a function of n .

18. Let $G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

be the group under matrix multiplication. Consider the following statements :

(A) G is a cyclic group of order 4 ;
(B) G is a non-cyclic group of order 4 ;
(C) G has 3 subgroups of order 2 .

Which of the following is valid :

(a) A only ; (b) B only ; (c) A, B, C ;
(d) B, C ; (e)* A, C .

19. A man stands at the origin of the x-axis and tosses a balanced die. After each toss, he takes one step in the positive direction if the die shows 1 or 2; he takes one step in the negative direction if the die shows 3 or 4; and he remains stationary if the die shows 5 or 6. The probability that the man remains at the origin after five tosses is

(a)* $\frac{17}{81}$; (b) $\frac{1}{3}$; (c) $\frac{1}{243}$; (d) $\frac{50}{243}$; (e) $\frac{11}{32}$

20. Let complex numbers α, β, γ be the roots of the equation $x^3 - 1 = 0$. Denote the set $\{a\alpha + b\beta + c\gamma \mid a,b,c \in \mathbb{R}\}$ by W. Then

(a) W is a real vector space of dimension 3 ;
(b)* W is a real vector space of dimension 2 ;
(c) W is a real vector space of infinite dimension ;
(d) W is a real vector space of finite dimension which cannot be determined ;
(e) W is not a real vector space .

<u>Part 2</u>

Saturday, 16 April 1977 3.00p.m. - 5.00p.m.

1. Let f be a function such that

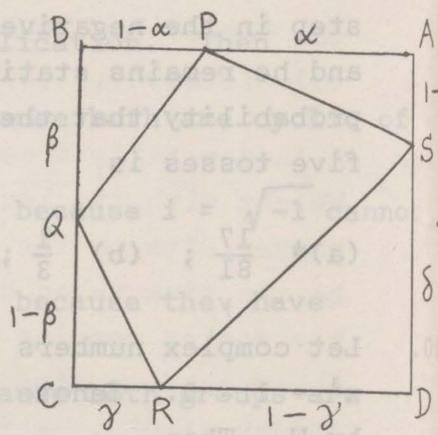
$$f(x_1 + x_2) = f(x_2) + f(x_2)$$

for all real numbers x_1 and x_2. Suppose that f is differential at some point x_0. Prove that

$$f(x) = \alpha x$$

for some constant α.

2. Let PQRS be a quadrangle inscribed in a square ABCD of unit area, as shown in the diagram. Denote the lengths of AP, BQ, CR and DS by α, β, γ and δ respectively.



(i) If PQRS is a parallelogram, show that $\alpha = \gamma$ and $\beta = \delta$.

(ii) If PQRS is a parallelogram of minimal perimeter among all the inscribed parallelograms, show that

$$\alpha + \delta = 1, \quad \beta + \gamma = 1$$

and PQRS is a rectangle.

3. (i) Define

$$\max(x,y) = \begin{cases} x & \text{if } y \leq x \\ y & \text{if } x < y . \end{cases}$$

Show that, if α and β are positive and $\alpha + \beta = 1$, then

$$\alpha x + \beta y \leqslant \max(x,y) .$$

(ii) Show that

$$\frac{a + c}{b + d} \leqslant \max\left(\frac{a}{b}, \frac{c}{d}\right)$$

where a, b, c and d are positive.

(iii) Find a condition under which

$$\frac{a + c}{b + d} = \max\left(\frac{a}{b}, \frac{c}{d}\right) .$$

4. Prove that $3.5^{2n+1} + 2^{3n+1}$, where n is a non-negative integer, is divisible by 17, but not by 7.

5. (i) Let α, β and γ be the roots of the equation $x^3 - 3x + 1 = 0$. Show that β and γ can be expressed as polynomials in α.

(ii) Give an example of cubic polynomial equation with roots α, β and γ such that β and γ cannot be expressed as polynomials in α.

6. A plumb-line, held vertically in a truck travelling along a horizontal path with constant acceleration f, is suddenly released. If θ is the angle of deviation of the plumb-line from the vertical, obtain a differential equation for θ. Show that the maximum angle of deviation θ_{max} is given by

$$\theta_{max} = 2\tan^{-1}\left(\frac{f}{g}\right) .$$

7. Let G be a finite group and H a subgroup of G. The index of H in G, denoted by (G : H), is defined to be

$$(G : H) = \frac{\text{number of elements of } G}{\text{number of elements of } H} .$$

The subgroup generated by a subset S of G is the subgroup consisting of all elements of the form $x_1^{n_1}x_2^{n_2} \ldots x_k^{n_k}$, where the x_i are in S, the n_i are integers and k is any positive integer.

If H is a subgroup of a finite group G with index (G : H) = 3 and if $a \in G$, $a \notin H$, show that the subgroup generated by the set

$$aH = \{ah \mid h \in H\}$$

is exactly G.

8. A cowboy unfortunately calls a Professor of Mathematics to a gun duel. Conscious of the inherent inferiority of the Professor in such a game, he invites the Professor to lay down his own terms and conditions. The following conditions are agreed upon.

The duellists will use similar revolvers with bullets travelling with velocity v, with the Professor standing at the centre of a roundabout of radius R which is capable of revolving with angular velocity ω (say) in the counterclockwise direction. The cowboy is required to position himself at the edge of the revolving roundabout. The Professor has the option to choose the angular velocity. How should each person aim in order to hit the opponent? Discuss three special cases when ω is such that

(i) ωR is small compared to v (ii) $\omega R = v$ and

(iii) $\omega R > v$, and show that the Professor has indeed placed himself in a favourable position.

9. Let a, b, c be the sides of a right -angled triangle where a, b, c are integers. Prove that 60 is a factor of abc. (Example : a = 5, b = 4, c = 3).

10. Suppose there are $n(\geqslant 2)$ urns each containing r black balls and r white balls. A ball is drawn at random from the 1st urn and put into the 2nd urn. A ball is then drawn from the 2nd urn and put into the 3rd urn. This process is continued until a ball is drawn from the nth urn and put back into the first urn. Show that the probability that the composition of balls in the 1st urn remains unchange is

$$\frac{1}{2}\left\{1 + \left(\frac{1}{2r+1}\right)^{n-1}\right\} .$$

Solutions to Part 2

1. Let x be any real number. Then

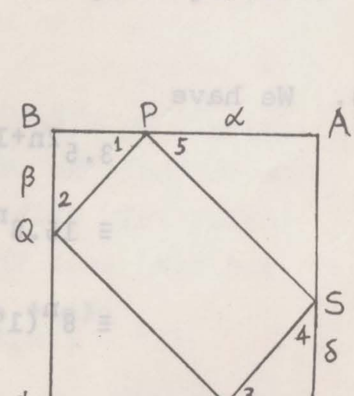
$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x) + f(h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

$$= f'(x_0)$$

Hence $f'(x) = f'(x_0)$ for all $x \in R$, and so $f(x) = f'(x_0)x + C$, where C is a constant. Put $x = 0$: $0 = f(o) = f'(x_0) \cdot 0 + C$. Therefore $f(x) = f'(x_0)x$ for all $x \in R$.

2. (i) If PQRS is a parallelogram, then $\angle 1 = \angle 3$, $\angle 2 = \angle 4$ and $PQ = RS$. Hence $\Delta BPQ \cong \Delta DRS$, and $\beta = \delta$, $1-\alpha = 1-\gamma$.



(ii) Hold Q (and S) fixed for the time being. The resulting parallelogram with varying P (and R) is of minimal perimeter if

$$\frac{PQ}{PS} = \frac{\beta}{1-\delta} = \frac{\beta}{1-\beta} .$$

But then the right-angled triangles BPQ and ASP are similar and

$$\frac{\beta}{1-\beta} = \frac{PQ}{PS} = \frac{BP}{PA} = \frac{1-\alpha}{\alpha} .$$

Hence $\alpha+\beta = 1$. By (i), $1 = \alpha+\beta = \alpha+\delta = \beta+\gamma$. It follows, also $\angle 1 = \angle 2 = 45^o$. Similarly $\angle 5 = 45^o$ and PQRS is a rectangle.

3. (i) Summing the inequalities

$$\alpha x \leqslant \alpha \max(x,y) ,$$

$$\beta y \leqslant \beta \max(x,y) ,$$

We have the required result.

(ii) Now $\quad \frac{a+c}{b+d} = \frac{b}{b+d} \cdot \frac{a}{b} + \frac{d}{b+d} \cdot \frac{c}{d}$.

Use (i) with $\alpha = \frac{b}{b+d}$, $\beta = \frac{d}{b+d}$, $x = \frac{a}{b}$, $y = \frac{c}{d}$.

(iii) Equality holds when $\frac{a}{b} = \frac{c}{d}$.

4. We have

$$3.5^{2n+1} + 2^{3n+1} = 15\cdot(25)^n + 2.8^n$$

$$\equiv 15.8^n + 2.8^n \pmod{17}$$

$$\equiv 8^n(17) \equiv 0 \pmod{17}.$$

Also,

$$15 \cdot (25)^n + 2.8^n \equiv 1.4^n + 2.8^n \pmod 7$$

If $3.5^{2n+1} + 2^{3n+1} \equiv 0 \pmod 7$,

then $4^n(1 + 2^{n+1}) \equiv 0 \pmod 7$.

Since $4^n \not\equiv 0 \pmod 7$, it follows that $1 + 2^{n+1} \equiv 0 \pmod 7$.

This is clearly not true for n = 1,2,3.

This problem may also be solved by induction.

5. (i) The discriminant of the cubic equation $x^3 + px + q = 0$ is given by

$$D = -4p^3 - 27q^2$$

$$= [(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)]^2,$$

where α, β, γ are the roots of the above cubic equation.

For the equation $x^3 - 3x + 1 = 0$, we have

$$D = 9^2 = (\beta-\gamma)^2(\alpha-\beta)^2(\alpha-\gamma)^2 .$$

Now $(\alpha-\beta)^2(\alpha-\gamma)^2 = [\alpha^2 - (\beta+\gamma)\alpha + \beta\gamma]^2$

$$= 9(\alpha^2-1)^2$$

since $\alpha+\beta+\gamma = 0, \quad \alpha\beta\gamma = -1$.

Hence $(\beta-\gamma)^2 = \left(\frac{3}{\alpha^2-1}\right)^2$.

Let $\frac{3}{\alpha^2-1} = a\alpha^2 + b\alpha + c.$

Then $3 = (\alpha^2-1)(a\alpha^2 + b\alpha + c)$

$$= (2a + c)\alpha^2 + (2b - a)\alpha - (b + c),$$

in view of the relations

$$\alpha^3 = 3\alpha - 1, \quad \alpha^4 = 3\alpha^2 - \alpha .$$

Equating coefficients, we have

$$2a + c = 0 ,$$

$$2b - a = 0 ,$$

$$b + c = -3 .$$

Solving, we have $a = 2, b = 1, c = -4$.

$$\text{Thus} \left(\frac{3}{\alpha^2 - 1}\right)^2 = (2\alpha^2 + \alpha - 4)^2$$

$$= 12 - 3\alpha^2 \ldots\ldots (*)$$

$$\text{Moreover, } (\beta - \gamma)^2 = (\beta + \gamma)^2 - 4\beta\gamma$$

$$= \alpha^2 - 4(\alpha^2 - 3)$$

$$= 12 - 3\alpha^2 .$$

Taking the positive square root in (*), we have

$$\beta - \gamma = 2\alpha^2 + \alpha - 4 .$$

But $\beta + \gamma = -\alpha$.

Therefore, $\beta = \alpha^2 - 2$,

$$\gamma = -\alpha^2 - \alpha + 2 .$$

Note. An alternative way of finding $1/(\alpha^2 - 1)$ is to note that the polynomials $x^2 - 1$ and $x^3 - 3x + 1$ are relatively prime, so that there are polynomials $a(x)$ and $b(x)$ such that

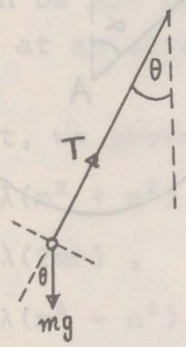
$$(x^2 - 1)a(x) + (x^3 - 3x + 1)b(x) = 1 .$$

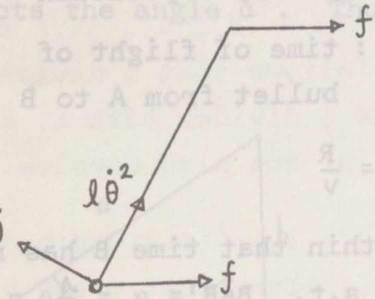
Putting $x = \alpha$, we have

$$(\alpha^2 - 1)a(\alpha) = 1 ,$$

and a(x) is obtained by the usual Euclidean algorithm.

6.





Force diagram   Acceleration diagram

If the length of the string is ℓ, and the mass is m, we have mg $\sin\Theta = m(f \cos\Theta - \ell\ddot{\Theta})$,

i.e. $\quad g \sin\Theta = f \cos\Theta - \ell\ddot{\Theta}$

Integrating, we have

$$g(1 - \cos\Theta) = f \sin\Theta + \frac{\ell\ddot{\Theta}^2}{2}$$

At Θ_{max}, $\dot{\Theta} = 0$. Hence

$$\tan\left(\frac{\Theta_{max}}{2}\right) = f/g$$

$$\Theta_{max} = 2 \tan^{-1}(f/g) .$$

7. Let K be the subgroup of G generated by aH. Then $aH \subseteq K$ and $H \subseteq K$ because $(a.1)^{-1}(aH) \subseteq K$.

Hence $\qquad (K : H) > 1 .$

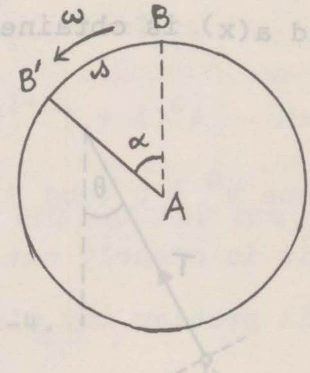
But

$$3 = (G : H) = (G : K)(K : H) .$$

Therefore $\qquad (K : H) = 3.$

This means that $(G : K) = 1$, i.e. $G = K$.

8. A : Professor's position



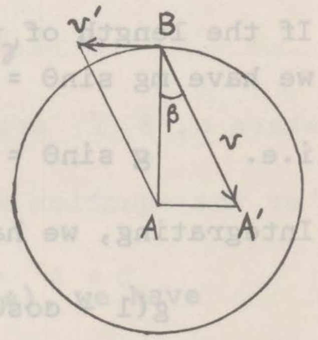
B : Cowboy's position

t : time of flight of bullet from A to B

$$t = \frac{R}{\nu}$$

Within that time B has moved to B' s.t. $B\hat{A}B' = \alpha = \frac{s}{R} = \omega t = \frac{\omega R}{\nu}$.

I. Hence A must fire in the direction AB' at an angle α to AB.



II. For B (Cowboy), he will have a velocity $\nu' = \omega R$. To shoot the opponent, he must aim along BA', so that the resultant of ν and ν' is along BA

$$\sin\beta = \frac{\omega R}{\nu}$$

(i) $\omega R << \nu$, $\beta \simeq \frac{\omega R}{\nu} = \alpha$.

In this case, both should aim at the same angle to the left of their respective opponents.

(ii) When $\omega R = \nu$, $\beta = \frac{\pi}{2}$.

Time of flight of the bullet, $t' = \frac{R}{\nu \operatorname{Cos}\beta} = \frac{R}{\nu}\frac{1}{\sqrt{\left(1 - \frac{\omega^2 R^2}{\nu^2}\right)}}$

$$= \frac{1}{\sqrt{\left(\frac{\nu^2}{R^2} - \omega^2\right)}} .$$

When $\omega R = \nu$, $t' = \infty$. Hence B cannot hit A.

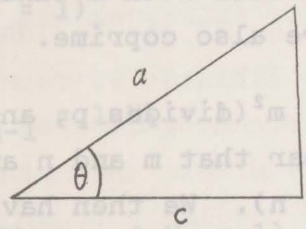
(iii) When $\omega R > \nu$, the resultant velocity cannot be directed towards A. Thus A cannot be hit by B. But A can be hit by B, if he selects the angle α. Thus A is at an advantage over B.

9. First, we prove that

$a = \lambda(m^2 + n^2)$,

$b = \lambda(2mn)$,

$c = \lambda(m^2 - n^2)$,



for some integers λ, m, n.

Let $\tan\frac{1}{2}\theta = t$, so that $\tan\theta = \dfrac{2t}{1-t^2}$, $\sin\theta = \dfrac{2t}{1+t^2}$.

Therefore

$$\frac{a}{1+t^2} = \frac{b}{2t} = \frac{c}{1-t^2} \quad \ldots. \quad (*)$$

We claim that t is a rational number. From (*), we have

$$bt^2 + 2ct - b = 0,$$

or

$$(bt + c)^2 - (b^2 + c^2) = 0,$$

since we may assume $b \neq 0$. But $a^2 = b^2 + c^2$. Hence

$$t = (a - c)/b\ ,$$

which is rational since a, b, c are integers. Thus we may set $t = n/m$, where m, n are relatively prime. From (*) we have

$$\frac{m^2 a}{m^2 + n^2} = \frac{m^2 b}{2mn} = \frac{m^2 c}{m^2 - n^2} = \mu\ ,$$

where $\mu = p/q$ for coprime integers p, q. Hence

$$m^2aq = p(m^2 + n^2) ,$$

$$m^2bq = p(2mn) ,$$

$$m^2cq = p(m^2 - n^2) .$$

It follows that q divides $m^2 + n^2$, 2mn and $m^2 - n^2$. Thus q divides both m^2 and n^2. Since m, n are coprime, m^2, n^2 are also coprime. Hence either q = 1 or q = 2.

If q = 1, m^2 divides p, and so we are done. If q = 2, it is clear that m and n are both odd. Put $x = \frac{1}{2}(m + n)$, $y = \frac{1}{2}(m - n)$. We then have

$$m^2a = p(x^2 + y^2) ,$$

$$m^2b = p(x^2 - y^2) ,$$

$$m^2c = p(2xy) .$$

Since m and n are coprime, m^2 and $x^2 + y^2 = \frac{1}{2}(m^2 + n^2)$ are also coprime. Hence m^2 divides p, and so the claim is proved.

Finally, it is sufficient to consider $a = m^2 + n^2$, $b = 2mn$, $c = m^2 - n^2$. We need only to show that 30 divides $mn(m^2 - n^2)(m^2 + n^2)$. It is clear that 2 divides $mn(m + n)$. Since $m \equiv \pm 1 \pmod 3$, $n \equiv \pm 1 \pmod 3$, it follows that 3 divides $m^2 - n^2$. Finally m^2, $n^2 \equiv 1$ or $4 \pmod 5$ implies that 5 divides $(m^2 - n^2)(m^2 + n^2)$.

10. Let X_i = 1 or 0 depending on whether a black ball or a white ball is drawn from the i^{th} urn. Then the required probability is

$$P(n,r) = P(X_n = 1, X_1 = 1) + P(X_n = 0, X_1 = 0)$$

$$= 2P(X_n = 1, X_1 = 1), \text{ by symmetry.}$$

If n = 2, then $P(n,r) = 2P(X_2 = 1 | X_1 = 1) \times P(X_1 = 1)$

$$= 2 \cdot \frac{r + 1}{2r + 1} \cdot \frac{1}{2} = \frac{r + 1}{2r + 1} .$$

For $n \geqslant 3$, we have

$$x_n = P(X_n = 1, X_1 = 1) = P(X_n = 1, X_{n-1} = 1, X_1 = 1)$$

$$+ P(X_n = 1, X_{n-1} = 0, X_1 = 1)$$

$$= P(X_n = 1 | X_{n-1} = 1) P(X_{n-1} = 1, X_1 = 1)$$

$$+ P(X_n = 1 | X_{n-1} = 0) P(X_{n-1} = 0, X_1 = 1)$$

$$= \frac{r + 1}{2r + 1} x_{n-1} + \frac{r + 1}{2r + 1} (\tfrac{1}{2} - x_{n-1})$$

$$= \frac{1}{2r + 1} x_{n-1} + \frac{1}{2(2r + 1)}$$

$$= \left(\frac{1}{2r + 1}\right)^{n-2} x_2 + \frac{r}{2(2r + 1)} \sum_{k=0}^{n=3} \left(\frac{1}{2r + 1}\right)^k$$

$$= \left(\frac{1}{2r + 1}\right)^{n-2} \frac{r + 1}{2(2r + 1)} + \frac{r}{2(2r + 1)} \frac{1 - \left(\frac{1}{2r + 1}\right)^{n-2}}{1 - \frac{1}{2r + 1}}$$

$$= \left(\frac{1}{2r + 1}\right)^{n-2} \frac{r + 1}{2(2r + 1)} + \frac{1}{4} \left[1 - \left(\frac{1}{2r + 1}\right)^{n-2} \right]$$

$$P(n, r) = \left(\frac{1}{2r + 1}\right)^{n-2} \frac{r + 1}{2r + 1} + \frac{1}{2} - \frac{1}{2} \left(\frac{1}{2r + 1}\right)^{n-2}$$

$$= \frac{1}{2} \left[1 + \left(\frac{1}{2r + 1}\right)^{n-1} \right] .$$