INEQUALITIES FOR SOME CYCLIC SUMS

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A sum of the form

$$\sum_{r=1}^{n} f(x_{r+1}, x_{r+2}, \ldots, x_{r+n}),$$

where $x_{s+n} = x_s$ for each s and n is a positive integer, is called a cyclic sum. If this sum is denoted by $F_n(x_1, x_2, \ldots, x_n)$ then it is clear that

$$F_n(x_{s+1}, x_{s+2}, \ldots, x_{s+n}) = F_n(x_1, x_2, \ldots, x_n)$$

for each s. It is because of this that the sum is called a cyclic sum.

In this paper we are concerned mainly with inequalities for the cyclic sum

$$S_n(x_1, x_2, \ldots, x_n) = \sum_{r=1}^{n} \frac{x_r}{x_{r+1} + x_{r+2}},$$

where $x_{s+n} = x_s \geq 0$ and $x_{s+1} + x_{s+2} > 0$ for each s.

It is trivial that $S_1(x_1) = \frac{1}{2}$ and $S_2(x_1, x_2) = 1$.

In 1903, A. M. Nesbitt [1] asked for a proof of the inequality

$$S_3(x_1, x_2, x_3) \geq \frac{3}{2}.$$

(Three known proofs of this are given in the appendix to this paper).

Over 50 years later, in 1954, H. S. Shapiro [2] asked for a proof of the inequality

$$S_n(x_1, x_2, \ldots, x_n) \geq \frac{n}{2}$$

for all positive integers n. At the present time it is known that this inequality is true for all $n \leq 10$ and false for all even $n \geq 14$ and all odd $n \geq 25$. For each of the remaining values of n (namely 11, 12, 13, 15, 17, 19, 21 and 23) it is not known whether the ineuqality is true or false.

H. S. Shapiro proved the inequality for n = 3 and n = 4 and C. R. Phelps for n = 5, but their proofs have not been published. L. J. Mordell [3] and the author [4] proved the inequality for $3 \leq n \leq 6$. Later D. Ž. Djokovic [5] proved the inequality for n = 8. Using this result, B. Bajšanski [6] and the author [7] proved that the inequality holds for n = 7 also. More recently, P. Nowosad [8] proved the inequality for n = 9 and 10.

M. J. Lighthill (see [9]) proved that the inequality is false for n = 20. He himself extended his method to prove the inequality false for n = 14. His proof has not been published. For n = 14, using Lighthill's method, A. Zulauf [10] and M. Herschorn and J. E. L. Peck [11] proved the same result. That the inequality is false for all even $n \geq 14$ follows from this, since it can be easily seen that

$$S_{n+2}(x_1,x_2,\ldots\ldots,x_n,x_1,x_2) = S_n(x_1,x_2,\ldots,x_n) + 1.$$

R. A. Rankin [12], using Lighthill's result for n = 20, proved that the inequality is false for all sufficiently large odd n. Later A. Zulauff [13] proved that the inequality is false for all odd $n \geq 53$. This was improved by the author [7], who proved that the inequality is false for all odd $n \geq 27$. Later, D. E. Daykin [14] and M. A. Malcolm [15] proved that the inequality is false for n = 25.

In his paper [12] Rankin also proved that there is a positive number $\lambda < \frac{1}{2}$ with the property that

$$S_n(x_1,x_2,\ldots,x_n) \geqslant \lambda n$$

is true for all n, but

$$S_n(x_1,x_2,\ldots,x_n) \geqslant (\lambda+\varepsilon)n$$

is not true for all n and all $x_1,x_2,\ldots,x_n$, however small $\varepsilon > 0$ is. He stated that he could prove that $\lambda > 0.3$. Later he [16] published a proof showing that $\lambda > 0.33$. The author proved that $\lambda > 0.45$ in [17] and later that $\lambda > 0.46$ in [18]. More recently V. G. Drinfeld [19] has proved that $\lambda = 0.494\ldots$.

In [4] and [20], the author investigated the inequality

$$T_n(x_1,x_2,\ldots,x_n) = \sum_{r=1}^{n} \frac{x_r}{x_{r+1}+x_{r+2}+\ldots+x_{r+m}} \geqslant \frac{n}{m},$$

where $x_{n+s} = x_s \geqslant 0$ and $x_{s+1} + x_{s+2} + \ldots + x_{s+m} > 0$ for each s, and proved that the inequality is true if

$n|m+2$ or $2m$ or $2m+1$ or $2m+2$,

or $n|m+3$ and $n = 8$ or 9 or 12,

or $n|m+4$ and $n = 12$.

For $m \geqslant 3$ it is not known whether there are any other (m,n) for which the inequality holds.

D. E. Daykin [14] considered the inequality

$$\sum_{r=1}^{n} \left(\frac{2x_r}{x_{r+1}+x_{r+2}}\right)^t \geqslant n,$$

where $x_{s+n} = x_s \geqslant 0$ and $x_{s+1} + x_{s+2} > 0$ for each s, and proved that it is true for $t \geqslant 2$. Using his method, the author [21] proved that the inequality is true for $t \geqslant \frac{\sqrt{5}+1}{2}$

= 1.6 ... . The smallest T such that the inequality is true for all n and all $t \geqslant T$ is not known.

Other related inequalities have also been studied by various authors (see, e.g., [3], [4], [14], [17], [18] and [20] to [24]). An expository account of cyclic inequalities, covering many of the publications up to 1968, is given in the book [25] by D. S. Mitronović.

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Appendix

Below are three proofs of the inequality

$$S_3(x_1,x_2,x_3) \geqslant \frac{3}{2}.$$

First proof. The inequality is equivalent to

$$\frac{x_1+x_2}{x_2+x_3} + \frac{x_3+x_1}{x_2+x_3} + \frac{x_2+x_3}{x_3+x_1} + \frac{x_1+x_2}{x_3+x_1} + \frac{x_3+x_1}{x_1+x_2} + \frac{x_2+x_3}{x_1+x_2} \geqslant 6 ,$$

which follows from the inequality between arithmetic and geometric means.

This proof can be generalized to prove that

$$S_n(x_1,x_2,\ldots,x_n) \text{ or } S_n(x_n,x_{n-1},\ldots,x_1) \geqslant \frac{n}{2} .$$

Second proof. The inequality is equivalent to

$$(x_1+x_2+x_3)\left(\frac{1}{x_2+x_3} + \frac{1}{x_3+x_1} + \frac{1}{x_1+x_2}\right) \geqslant \frac{9}{2} ,$$

which is true, since

$$\frac{1}{3}\left(\frac{1}{x_2+x_3} + \frac{1}{x_3+x_1} + \frac{1}{x_1+x_2}\right) \geqslant \frac{3}{2(x_1+x_2+x_3)}$$

by the inequality between arithmetic and harmonic means.

This proof can be generalized to prove that

$$T_{m+1}(x_1,x_2,\ldots,x_{m+1}) \geqslant \frac{m+1}{m} .$$

Third proof. Considering $S_3(x_1,x_2,x_3)$ as a weighted sum of $\frac{1}{x_2+x_3}$, $\frac{1}{x_3+x_1}$, $\frac{1}{x_1+x_2}$ with weights x_1, x_2, x_3, respectively, we can see that

$$S_3(x_1,x_2,x_3) \geqslant \frac{(x_1+x_2+x_2)^2}{x_1(x_2+x_3)+x_2(x_3+x_1)+x_3(x_1+x_2)}$$

by the inequality between weighted arithmetic and harmonic means. Hence the inequality $S_3 \geqslant 3/2$ follows if we can prove that the quadratic form

$$(x_1+x_2+x_3)^2 - \frac{3}{2} x_1(x_2+x_3) - \frac{3}{2}x_2(x_3+x_1) - \frac{3}{2}x_3(x_1+x_2)$$

is positive semi-definite. This is true since the quadratic form is equal to

$$(x_1-\tfrac{1}{2}x_2-\tfrac{1}{2}x_3)^2 + \tfrac{3}{4}(x_2-x_3)^2 .$$

This last proof is more complicated than either of the other two proofs. It can, however, be generalized to prove that

$$S_n(x_1,x_2,\ldots,x_n) \geqslant \frac{n}{2}$$

for $n = 4,5,6$ (see [4]), and also to prove all the known true cases of

$$T_n(x_1,x_2,\ldots,x_n) \geqslant \frac{n}{m}$$

for $m > 2$ (see [4], [20]).