A SURVEY OF THE STRUCTURE THEORY OF RINGS

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Introduction

The main aim of any structure theory of rings is to describe a ring in terms of some simpler ones. The first step in such a theory is to decide on the candidates for the 'simple' rings. The choice of 'simple' rings is not arbitrary. It depends very much on how decisive and how conceptually important are the results that can be produced. Once the class of 'simple' rings is decided on, it is necessary to find ways of putting these rings together to form a new class of rings, called the 'semi-simple'rings.

In a meaningful structure theory, it should not happen that every ring is 'semi-simple'. A ring may contain undesirable elements or ideals that get in our way of completely describing the ring in terms of 'simple' rings. The major effort in a structure theory is thus to decide on the desirable and undesirable properties that a ring may have in such a way that (i) all 'simple' rings have the desirable property; (ii) among all the ideals of a ring R with the undesirable property, there is a unique maximal one, say A; and (iii) the factor ring R/A is 'semi-simple'. Rings with the undesirable property are usually called 'radical' rings and the unique maximal ideal A, the 'radical' of R.

It is therefore evident that the trio (listing of desirable and undesirable properties of rings, selection of the class of 'simple' rings and identification of ways that 'simple' rings weave together to yield 'semi-simple' rings) are inter-related. If the structure theory is to be consequential, they will have to be considered simultaneously.

In every structure theory of rings, there are invariably three major components:

(i) The study of 'simple' and 'semi-simple' rings

(ii) The study of 'radical' rings

(iii) The construction of rings with a given 'radical' and 'semi-simple' factor ring.

In the existing structure theories, the first of the above problems has been handled most successfully, while little is known about the last problem. The purpose of this talk is to gather results on classes of 'simple' and 'semi-simple' rings that appear in the literature. A brief description of the radical theory of rings will also be given to illustrate the close tie between the trio mentioned above.

Radical Theory

Radical theory of rings was an area of fruitful mathematical research in 1940's and 1950's. Roughly speaking, it is a study of the undesirable properties of rings. There were various radical properties contemplated. In each case there resulted in a successful description of the corresponding 'semi-simple' rings in terms of 'simple' rings. These isolated instances conform to the general framework of a postulated system by the Russian mathematician V. Andrunakievic in 1958-61 [1,2].

A property P about rings is a radical property if the following conditions hold:

(A) Every homomorphic image of a ring with property P possesses the property P.

(B) Each ring R contains an ideal A with property P which contains every other ideal with property P.

(C) The different ring R/A is P-semi-simple, i.e., it does not contain any nonzero ideals with property P.

For brevity, a ring (resp. ideal) with property P will be called P-ring (resp. P-ideal). The unique maximal P-ideal of a ring is called the P-radical of the ring.

There are many ways radical properties can be constructed. We shall mention two: the upper radical property and the lower radical property determined by a class of rings.

A radical property can be constructed by specifying a class of semi-simple rings. In general, the class S of all semi-simple rings with respect to a radical property P satisfies the following condition:

(D) Every nonzero ideal of a ring of S can be mapped homomorphically onto some nonzero ring of S.

If we start a class X of rings which has (D), we can construct a radical property P(X) by defining a P(X)-ring to be one which has no nonzero homomorphic image R such that every nonzero ideal of R can be mapped homomorphically onto some nonzero·ring of X. This radical property is called the upper radical property determined by X.

A radical property can also be constructed by specifying a class of radical rings. Given a class of rings Z, we can enlarge it step by step through the following process and eventually arrive at a radical property. Define Z_1 to be the class of all rings which are homomorphic images of rings in Z. For any ordinal number ρ, if ρ is not a limit ordinal, Z_ρ is the class of all rings R such that every nonzero homomorphic image of R contains a nonzero ideal in $Z_{\rho-1}$. If ρ is a limit ordinal, then Z_ρ is the union of all Z_α $(\alpha < \rho)$. The union of all Z_ρ determines a radical property, called the lower radical property determined by Z.

Practically all of the significant radical properties that appeared in the literature can be represented as either upper radical properties or lower radical properties determined by some classes of rings. For example, the Baer lower radical B [3], the Levitzki radical L [4], the nil radical N [5], the Jacobson radical J [6] and the Brown-McCoy radical BM [7,8] can be aptly described in terms of upper and lower radical properties as follows:

B = lower re all nilpotent rings
= upper re all prime rings
L = lower re all locally nilpotent rings
= upper re all prime L-semi-simple rings
N = lower re all nil rings
= upper re all prime N-semi-simple rings
J = upper re all primitive rings
BM = upper re all simple rings with an identity

The corresponding radicals of a ring R can be written as intersection of specific types of ideals:

$B(R) = \bigcap$ all prime ideals of R
$L(R) = \bigcap$ all prime ideals P such that R/P is L-semi-simple
$N(R) = \bigcap$ all prime ideals P such that R/P is N-semi-simple
$J(R) = \bigcap$ all primitive ideals
$BM(R) = \bigcap$ all ideals M such that R/M is simple with an identity

A semi-simple ring can therefore be represented as a subdirect sum of specific classes of rings.

Readers interested in radical theory of rings may consult Divinsky [9].

Structure of Finite Dimensional Simple and Semi-simple Algebras

The pioneer work on the structure of finite dimensional simple algebras was probably due to J.H. MacLagan Wedderburn. In [10], Wedderburn proved the following celebrated theorem:

Theorem (Wedderburn 1908) Any simple finite dimensional algebra can be represented as the tensor product of a finite dimensional division algebra and a simple matrix algebra.

A semi-simple finite dimensional algebra can be represented uniquely as the direct sum of a finite number of simple finite dimensional algebras.

All algebras are to have an identity, simple means without any nonzero proper ideals and semi-simple means having no nonzero nilpotent ideals.

In essence, Wedderburn's work reduces the study of semi-simple and simple finite dimensional algebra to that of finite dimensional division algebras. Investigations into simple (and hence division) algebras had been carried out in the next couple of decades. Earlier in 1905, Wedderburn [11] established that a finite division algebra is a field. For division algebra D finite dimensional over its centre F, the closest result that we can get is that : if K is a maximal subfield of D, then $[D:K] = [K:F] = \sqrt{[D:F]}$. There is always a maximal subfield which is separable over the centre. Established in this period was also the following theorem which has many applications:

<u>Theorem</u> (Noether-Skolem) Let R be a simple Artinian ring with centre F and let A and B be simple subalgebras of R which contain F and are finite dimensional over it. If ρ is an F-isomorphism of A onto B, then there is an invertible element a of R such that $\rho(x) = a^{-1}xa$ for all x in A.

Return now to the structure theory of divisible algebra. By Wedderburn's theorem, if A and B are finite dimensional central simple algebras over the field F, then

$$A \cong C \otimes_F F_n \quad \text{and} \quad B \cong D \otimes_F F_m$$

where C and D are finite dimensional divisional algebras having F as centre and F_n denotes the ring of all nxn matrices over F. An equivalence relation can be defined on the collection $\mathcal{C}(F)$ of all central simple algebras over F when we set

$$A \sim B \text{ if } C \cong D \text{ (or equivalent } A \otimes_F F_m = B \otimes_F F_n).$$

The equivalence classes form an abelian group B(F), (called the Brauer group of F) under the operation induced by tensor product.

Finite dimensional central simple algebras can be constructed (up to equivalence) as follows:

Let $A = D \otimes_F F_k$ be a finite dimensional central simple algebra over F with D a division algebra; and let K be a separable maximal subfield of D. Then $[K:F] = \sqrt{[D:F]} = n$ say. Let L be the normal closure of K with $[L:K] = m$. Then $B = D \otimes_F F_m$ is central simple over F, $B \supseteq L$ and $[B:F] = [L:F]^2$. So A and B are equivalent.

Let G be the Balois group of L over F. By Noether-Skolem theorem, for each σ in G, there exists t_σ in B such that $\sigma(x) = t_\sigma x t^{-1}$ for all x in L. The set of all t_σ(σ in G) are linear independent over L. If σ and τ are two elements of G, then $0 \neq t_{\sigma\tau}^{-1} t_\sigma t_\tau = f(\sigma,\tau) \in L$. The triple (L,G,f) has the following properties:

(i) $B = \{\sum_{\sigma \in G} t_\sigma a_\sigma \mid a_o \in L\}$,

(ii) $a t_\sigma = t_\sigma \sigma^{-1}(a)$ for $a \in L$ and $\sigma \in G$,

(iii) $t_\sigma t_\tau = t_{\sigma\tau} f(\sigma,\tau)$ for $\sigma,\tau \in G$,

(iv) $f(\sigma,\tau\nu) f(\tau,\nu) = f(\sigma\tau,\nu)\nu^{-1}(f(\sigma,\tau))$.

Thus, up to equivalence, a finite dimensional central simple algebra A over R can be represented as the triple (L,G,f) where L is a finite normal separable extension of F, G is the Galois group of L over F and f is a mapping of $G \times G$ into $L^* = L \setminus \{0\}$ satisfying (iv). The algebra A consists of all linear combinations of the indeterminates t_σ($\sigma \in G$) with coefficients in L. The operations on A are governed by (ii) and (iii).

With these preparations, the following result can be proved.

Theorem. The Brauer group B(F) of the field F is torsion. In fact the order of the class of a central simple algebra A over F divides n, where $n^2 = [D:F]$, $A = D \otimes_F F_n$.

Structurally a finite dimensional central division algebra over F can be factored as a tensor product of a finite number of central division algebras of prime power degree over F:

<u>Theorem</u>. If D is a finite dimension central division algebra over F with $\sqrt{[D:F]} = p_1^{m_1} \ldots p_k^{m_k}$ where p_i are distinct primes, then

$$D = D_1 \otimes_F D_2 \otimes_F \ldots \otimes_F D_n$$

where D_i are central division algebra over F with $\sqrt{[D_i:F]} = p_i^{m_i}$.

There is a concise account of the structure of simple algebras given in Herstein [12].

To end this section, it may be interesting to mention the work of Herstein and Amitsur on finite subgroups of the multiplicative group of a division ring. Determination of all finite groups that can be embedded in a division ring was intiated by Herstein [13] and carried by Amitsur [14]. These groups are classified into five classes connected in some way to the finite groups of rotations of the 3-dimensional Encludean sphere. Main results are quoted in the following theorems:

<u>Theorem</u> (Herstein 1953) The only finite subgroup of a division ring of finite characteristic are cyclic.

Subroups of odd order of a division ring of characteristic zero are metacyclic: $\langle a,b \mid a^n = b^m = 1, bab^{-1} = a^r \rangle$.

<u>Theorem</u> (Amitsur 1955) Finite subgroups G of a division ring are of one of the following types:

(A) All Sylow subgroups of G are cyclic.

(B) The odd Sylow subgroups of G are cyclic and the even Sylow subgroup of G is a generalized quaternion group of order 2^r $(r \geqslant 3)$:

$$<a,b|a^{2^r} = b^2, b^4 = 1, bab^{-1} = a^{-1}>.$$

Theorem (Amitsur 1955) A finite group G can be embedded in a division ring if and only if G is of the following types:

(T1) Cyclic groups

(T2) D-group $G_{m,r} = <a,b|a^m = 1, b^n = a^t, bab^{-1} = a^r>$ where $(m,r) = 1$, $n = \min\{k|a^k = 1\ (m)\}$, $t = m/(r-1,m)$, m and r satisfy certain conditions.

(T3) T-group $T^* \times G_{m,r}$, where

(i) T* is the binary tetrahedral group: The centre C of T has two elements and T*/C is the tetrahedral group,

(ii) for any p|m, the minimal c_p satisfying $2^{c_p} = 1\ (p)$ is odd.

(T4) The binary octahedral group O*: The centre C of O* has two elements and O*/C is the octahedral group.

(T5) The binary icosahedral group I*: The centre C of I* has two elements and I*/C is the icosahedral group.

Structure of Artinian Rings

Right Artinian rings (rings satisfying the descending chain condition on right ideals) if simple or semi-simple have the similar structure as finite dimensional simple or semi-simple algebras. This was established by E. Artin [15].

Theorem (Wedderburn 1908, Artin 1927) A semi-simple right Artinian ring is the direct sum of a finite number of simple right Artinian rings.

Each simple right Artinian ring is isomorphic to D_n, the ring of all $n \times n$ matrices over the division ring D. Moreover n is unique, as is D up to isomorphism.

There is an interesting structure for the additive group of an Artinian ring. The study of the additive structure of rings was first carried out by L. Fuchs. One of the illustrious classes is the Artinian rings. A perfect structure on the additive groups of such rings is given in the following

Theorem (Szele-Fuchs [16]) In order that a group A be the additive group of an Artinian ring, it is necessary and sufficient that it has the form

$$A = \bigoplus_{\mathfrak{m}} Q \quad \bigoplus_{\text{finite}} Z(p_i) \bigoplus_{\mathfrak{n}} Z(p_j^{e_j}), \; p_j^{e_j} | m$$

where $\mathfrak{m}$, $\mathfrak{n}$ are arbitrary cardinals, p_i, p_j are primes and m a fixed integer.

Structure of Noetherian Prime and Semi-prime Rings

The development of theory of right Noetherian rings (= rings with the ascending chain condition on right ideals) is one of the most recent mathematical activities. Noetherian rings are as a rule more difficult to deal with than Artinian rings as the former constitute a wider class than the latter in some sense. Precisely,

Theorem. A right Artinian ring R is right Noetherian

(1) if R has a right identity (Hopkins [17]).

(2) if and only if R contains no quasi-cyclic subgroups (Fuchs [18]).

(3) if and only if the annihilator of R is finite.

A.W. Goldie [19,20] has recently several renowned theorems on prime and semi-prime rings satisfying certain chain conditions on some of the right ideals. These results give penetrating information about the nature of such rings. Quite independently, L. Lesieur and R. Croisot [21] published similar results at about the same time.

Theorem (Goldie, Lesieur-Croisot) The following properties of ring R are equivalent:

(1) R is a right order in a semi-simple right Artinian ring.

(2) R is a semi-prime, right Goldie ring.

(3) A right ideal of R is essential if and only if it contains a regular element.

Corollary. A ring R is a right order in a simple right Artinian ring if and only if it is a prime, right Goldie ring.

Notes: (1) An element a of a ring R is regular if it is neither a left nor a right divisor of zero.

(2) A ring Q is a right quotient ring of R (and R is a right order in Q) if (i) every regular element of R has an inverse in A, and (ii) every element of Q is of the form ab^{-1} where a and b are elements of R with b regular.

(3) A right ideal A of R is a right annihilator if there is a subset S of R such that $A = \{x \in R | Sx = 0\}$.

(4) R is a right Goldie ring if (i) R satisfies the ascending chain condition on right annihilators, and (ii) R is of finite right rank, i.e., R contains no infinite direct sums of right ideals.

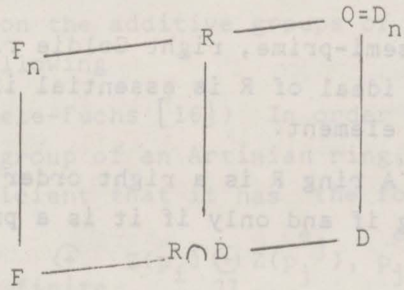
(5) A right ideal A of R is essential if $A \cap B \neq 0$ for all nonzero right ideal B of R.

A refinement of the Goldie-Lesieur-Croisot theorem is the following result of Faith-Utumi:

Theorem (Faith-Utumi [22]) Let R be a ring with a simple, right Artinian quotient ring Q. Then Q contains a complete set $M = \{e_{ij}\}$ of matrix units with the following property:

If D is the centralizer of M in Q, then R contains a subring $F_n = \sum_{i,j} Fe_{ij}$, where F is a right Ore domain (= integral domain all of whose nonzero right ideals are essential) contained in $R \cap D$ and D is the quotient division ring of F.

Diagrammetrically, the above theorem can be represented as follows:



All subrings of Q containing F_n are right orders of Q. The above theorem tells us explicitly how to obtain right orders of a simple, right Artinian ring.

Several lines of investigation on Noetherian rings diverge from the masterpiece of Goldie. Major ones are those on orders, localization and quotient rings. There is also a fine piece of work of Faith and others on simple Noetherian rings.

L. Small [23] has shown that a right Noetherian ring R is an order in a right Artinian ring if and only if the set of all regular elements of R coincides with the set of all c such that $cx \in N$ or $xc \in N \Rightarrow x \in N$, where N is the largest nilpotent ideal of R. This result does not assume semi-prime-ness of R or simplicity of the quotient ring.

Among the right orders in a ring Q, we may define an equivalence relation as follows: Two right orders R and S of Q are equivalent if there exist invertible elements a,b,c,d of Q such that $aRb \subseteq S$ and $cSd \subseteq R$. A maximal order is a right order R which is maximal (under set inclusion) in the equivalence class of R.

If R is a right order of Q, a right R-submodule I of Q is a fractional R-ideal if $aR \geq I \geq bR$ for some invertible elements a, b of Q. If $I \subseteq R$, then I is called an integral R-ideal.

Asano [24] studied a type of right orders R with an identity and for which all the fractional R-ideals in Q form a group under multiplication induced by that of Q. These right orders were named subsequently after him. Asano's work has recently been extended by Robson [25], Michler [26], Lenagan [27] and others. We shall mention two of the main theorems.

Theorem (Asano, Robson) Let R be a right order with an identity. Then the following are equivalent:

(1) R is Asano.

(2) R is a maximal order and integral R-ideals are R-projective.

(3) Every integral R-ideal is invertible.

Theorem (Asano, Michler, Lenagan) If a bounded prime right Noetherian ring is an Asano order, then it is a left order and is right and left hereditary.

Here a ring is bounded if every one sided nonzero ideal contains a nonzero ideal. It is right hereditary if every right ideal is projective.

As for the right Noetherian simple ring, Faith [28] showed in 1964 that if R is a (right Noetherian) simple ring with identity and with a uniform ideal U (= every nonzero right ideal in U is essential), then

(i) the ring K of all R-endomorphisms of U is a right Ore domain,

(ii) U, considered as a left K-module, is torsion free (of finite rank), and

(iii) R is isomorphic to the ring of all K-endomorphisms of U.

Hart [29] extended Faith's result by showing that for a simple ring R with an identity,

(a) R has a uniform right ideal if and only if R is of finite right rank.

(b) if R has a uniform right ideal U, then the ring K of all R-endomorphisms of U is a right Ore domain and R is isomorphic to $eK_n e$, where e is an idempotent in K_n.

(c) a uniform right ideal U of R is projective if and only if K is simple.

(d) if U is a projective uniform right ideal of R, then R is right Noetherian $\Longleftrightarrow$ K is right Noetherian.

Faith [30] subsequently elaborated the results and obtain the following three equivalent conditions on a ring R.

(1) R is a simple ring with a uniform right ideal U.

(2) R is isomorphic to the ring of all K-endomorphisms of U, where K is a right Ore domain with at most one non-trivial proper ideal, and U, as a left K-module, is finitely generated and projective.

(3) R is isomorphic to the ring of K_o-endomorphisms of U, where K_o is a right Ore domain with precisely three ideals and U, as K_o-module, is finitely generated and projective.

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