# ARE YOUR STUDENTS MATHEMATICALLY GIFTED?

Teh Hoon Heng
Nanyang University

## Introduction.

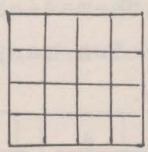
There are generally speaking two important tasks for every good mathematical teacher. The first is to help his students, even the weaker ones, to pass examinations. The second, however, is to discover the more talented students and inspire them to fully develop their potentialities.

From the past we see that most of the mathematicians had displayed their talents even when they were very young. Hence it is important for primary school teachers to know what the qualities of mathematically gifted students are and how to discover and inspire such students.

## The qualities of mathematically gifted students.

I The first aspect of mathematical talents is usually the ability to discover "patterns". The following example provides a test for this type of ability for children from Primary I to Primary III :

First draw a big square and then divide it into 16 smaller squares :



Next fill these small squares with integers 1, 2, 3, .., 16 as follows:

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 |

Fig. 1

Then take out the numbers in the two diagonals and then fill them back in the reverse order as shown below:

| | | | |
|---|---|---|---|
| 16 | 2 | 3 | 13 |
| 5 | 11 | 10 | 8 |
| 9 | 7 | 6 | 12 |
| 4 | 14 | 15 | 1 |

Fig. 2

The children are then asked to find the sums of the four numbers in every row and every column. Next, they are asked to find the sum of the four numbers in each of the two diagonalsas well as the sum of the four numbers in the corners of the square. To their surprise and excitement they find that all the sums are equal to 34.

We then ask the children the following question: "Can you discover other sums of four numbers giving also 34?" Children who are mathematically gifted will very quickly discover on their own many other such sums. The following gives 40 of them.

| | | | |
|---|---|---|---|
| A | A | A | A |
| B | B | B | B |
| C | C | C | C |
| D | D | D | D |

| | | | |
|---|---|---|---|
| A | C | C | B |
| D | A | B | D |
| D | B | A | D |
| B | C | C | A |

| | | | |
|---|---|---|---|
| A | A | B | B |
| A | A | B | B |
| C | C | D | D |
| C | C | D | D |

| | | | |
|---|---|---|---|
| A | A | B | B |
| C | C | D | D |
| D | D | C | C |
| B | B | A | A |

| | | | |
|---|---|---|---|
| A | C | D | B |
| B | D | C | A |
| A | C | D | B |
| B | D | C | A |

| | | | |
|---|---|---|---|
| A | B | C | D |
| A | B | C | D |
| A | B | C | D |
| A | B | C | D |

| | | | |
|---|---|---|---|
| A | C | D | A |
| C | B | B | D |
| D | B | B | C |
| A | D | C | A |

| | | | |
|---|---|---|---|
| A | C | D | B |
| D | B | A | C |
| C | A | B | D |
| B | D | C | A |

| | | | |
|---|---|---|---|
| A | C | D | B |
| A | C | D | B |
| B | D | C | A |
| B | D | C | A |

| | | | |
|---|---|---|---|
| A | B | A | B |
| C | D | C | D |
| D | C | D | C |
| B | A | B | A |

(In each of the above squares the sum of the four numbers denoted by the same letter is 34.)

Children who can discover more than 30 such sums are considered to be very good. Only gifted children can discover all the 40.

II The second aspect of mathematical talents is the ability to discover relationships between a new pattern and an old pattern. The following example provides such a test for children from Primary IV to Primary VI.

Take a piece of paper. Draw a big square and then divide it into 9 smaller squares. Then fill the small squares with integers 1, 2, 3...9 as follows :

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

By rotating the paper we obtain the following tables :

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

| | | |
|---|---|---|
| 3 | 6 | 9 |
| 2 | 5 | 8 |
| 1 | 4 | 7 |

| | | |
|---|---|---|
| 9 | 8 | 7 |
| 6 | 5 | 4 |
| 3 | 2 | 1 |

| | | |
|---|---|---|
| 7 | 4 | 1 |
| 8 | 5 | 2 |
| 9 | 6 | 3 |

By writing the numbers properly and exchanging the positions of the third and fourth tables we get:

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

(table 1)

| | | |
|---|---|---|
| 3 | 6 | 9 |
| 2 | 5 | 8 |
| 1 | 4 | 7 |

(table 3)

| | | |
|---|---|---|
| 7 | 4 | 1 |
| 8 | 5 | 2 |
| 9 | 6 | 3 |

(table 7)

| | | |
|---|---|---|
| 9 | 8 | 7 |
| 6 | 5 | 4 |
| 3 | 2 | 1 |

(table 9)

We then ask the children to study these four tables very carefully and then compare them with the multiplication tables of 1, 3, 7, and 9 which may be written as follows :

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

| | | |
|---|---|---|
| 3 | 6 | 9 |
| 12 | 15 | 18 |
| 21 | 24 | 27 |

| | | |
|---|---|---|
| 7 | 14 | 21 |
| 28 | 35 | 42 |
| 49 | 56 | 63 |

| | | |
|---|---|---|
| 9 | 18 | 27 |
| 36 | 45 | 54 |
| 63 | 72 | 81 |

The brighter children will then discover that the number patterns in tables 1, 3, 7, 9 are very similar to those of the multiplication tables of 1, 3, 7, 9. The only missing parts in the tables 1, 3, 7, 9 are the tens digits of some numbers. We then ask the children the following questions : "Can you discover a simple rule which will tell you how to put in the tens digits of some numbers in tables 1, 3, 7, 9 so as to obtain the multiplication tables of 1, 3, 7, 9?" This is a difficult problem. But if you have really mathematically gifted children in your class, they should be able to discover such a rule.

On the other hand if we start from the following pattern :

| | | |
|---|---|---|
| 2 | 4 | 6 |
| 8 | 0 | 2 |
| 4 | 6 | 8 |

We can then obtain the following tables :

(table 2)

| | | |
|---|---|---|
| 2 | 4 | 6 |
| 8 | 0 | 2 |
| 4 | 6 | 8 |

(table 4)

| | | |
|---|---|---|
| 4 | 8 | 2 |
| 6 | 0 | 4 |
| 8 | 2 | 6 |

(table 6)

| | | |
|---|---|---|
| 6 | 2 | 8 |
| 4 | 0 | 6 |
| 2 | 8 | 4 |

(table 8)

| | | |
|---|---|---|
| 8 | 6 | 4 |
| 2 | 0 | 8 |
| 6 | 4 | 2 |

They are very similar to the multiplication tables of 2, 4, 6 and 8.

| | | |
|---|---|---|
| 2 | 4 | 6 |
| 8 | 10 | 12 |
| 14 | 16 | 18 |

| | | |
|---|---|---|
| 4 | 8 | 12 |
| 16 | 20 | 24 |
| 28 | 32 | 36 |

| | | |
|---|---|---|
| 6 | 12 | 18 |
| 24 | 30 | 36 |
| 42 | 48 | 54 |

| | | |
|---|---|---|
| 8 | 16 | 24 |
| 32 | 40 | 48 |
| 56 | 64 | 72 |

There is also a simple rule to put in the missing tens digits of some numbers in tables 2, 4, 6, 8 so as to obtain the multiplication tables of 2, 4, 6 and 8. The really gifted children will discover that the following two very simple patterns

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

| | | |
|---|---|---|
| 2 | 4 | 6 |
| 8 | 0 | 2 |
| 4 | 6 | 8 |

in fact contain the multiplication tables of 1, 2, 3, 4, 6, 7, 8, 9. And since 5x8 = 8x5, 5x4 = 4x5,..., etc, the multiplication table of 5 (except the product 5x5) is also contained in the above two simple tables 1 and 2.

III The third aspect of mathematical talents is the ability to transform a problem in such a way that a nice pattern can be easily discovered. The following example is used to test students in Secondary I after they have learnt binary numbers.

Problem : A chess board consists of 64 squares. Suppose that we place one stone on the first square, 2 stones on the second and 4 stones on the third and so on... . How many stones are there altogether on the chess board?

In other words, the students are asked to find the sum

$$1 + 2 + 4 + 8 + \ldots + 2^{63} = ?$$

Students who are mathematically gifted may realize that this problem becomes a very easy problem if he writes the numbers in the binary notation. The above sum becomes:

$$1_{(2)} + 10_{(2)} + 100_{(2)} + \ldots + 1\overbrace{0\ldots\ldots\ldots\ldots0}^{63\ \text{zeros}}{}_{(2)} = ?$$

The answer is simply $\overbrace{1\,1\,1\,1\,\ldots\,1}^{64 \text{ terms}}{}_{(2)}$ .

Mathematically gifted students have also no difficulty in generalizing the method to find similar sums such as :

$$1 + 3 + 9 + 27 + \ldots + 3^{63} = ?$$

$$1 + 5 + 25 + 125 + \ldots + 5^{63} = ?$$

IV Another important aspect of mathematical talents is the ability to think logically together with the ability to understand and to appreciate a beautiful proof. For example, Secondary IV students are familiar with Pythagoras' Theorem. In order to test whether there are gifted students, the teacher may try to use the following proof.

THEOREM : Let A B C be a right angled triangle. Prove that the area of the square C B F E is equal to the sum of the area of the square A B Y X and the area of the square A C G H, (fig. 3).

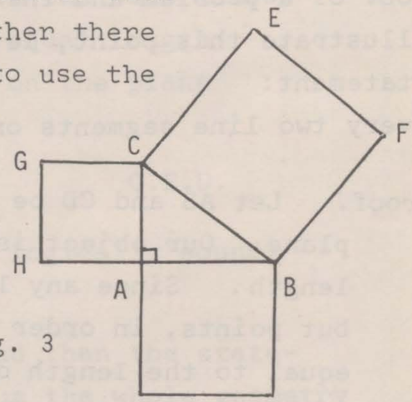



Fig. 3

Proof. Let A D ⊥ B C (fig. 4). Then we see that the three triangles A B C, D B A, D A C are similar. We also see that area A B C = area A B D + area A D C. i.e. I = II + III (fig. 5).

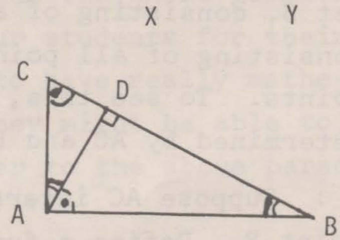
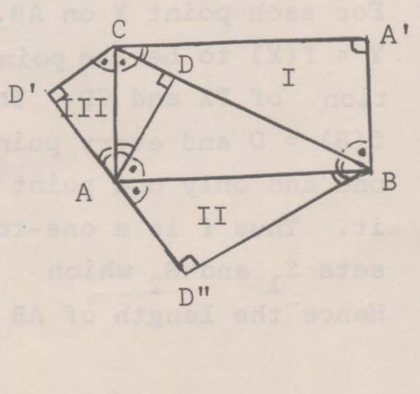
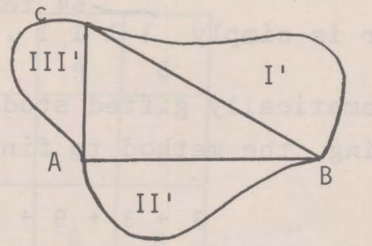



Fig. 4




This implies that if we construct any similar types of figures on the sides BC, CA and AB, the ratios between I' to I, II' to II and II' to III must be equal to a constant K, say.

Thus




I' = KI = K(II + III)
= KII + KIII
= II' + III'

Hence I' = II' + III'.

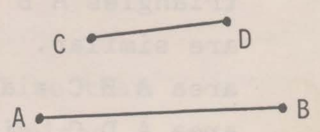
Thus Pythagoras' Theorem is simply a special case of this more general result.

Q.E.D.

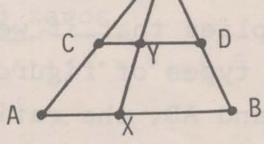
V Among all aspects of mathematical talents, perhaps the most difficult to find is the ability to see the root of a problem and the ability to solve it. To illustrate this point, let us consider the following statement:

Every two line segments on the plane have equal length.

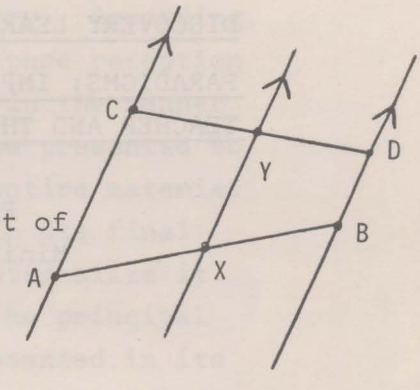
Proof. Let AB and CD be any two given line segments on the plane. Our object is to prove that they have equal length. Since any line segment consists of nothing but points, in order to show that the length of AB is equal to the length of CD we need only to show that the set $S_1$ consisting of all points on AB and the set $S_2$ consisting of all points on CD have the same number of points. To see this, let us consider the straight lines determined by AC and BD. There are two possible cases:




Case I : Suppose AC intersects BD at a point P. Define a function f from the set $S_1$ to the set $S_2$ as follows: For each point X on AB, we define Y = f(X) to be the point of intersection of PX and CD. Thus f(A) = C, f(B) = D and every point on AB has one and only one point associated with it. Thus f is a one-to-one correspondence between the sets $S_1$ and $S_2$ which have equal number of points. Hence the length of AB is equal to the length of CD.



Case 2. Suppose AC and BD are parallel. Define a function f from $S_1$ into $S_2$ as follows: For each point X on AB we define Y = f(X) to be the point of intersection of CD and the straight line passing through X and parallel to AC.




Then in the same manner we see that f is a one-to-one correspondence between the two sets $S_1$ and $S_2$. Thus AB and CD contain the same number of points. Hence they are of equal length. Hence every two line segments on the plane have equal length.

Q.E.D.

Problem : Is the proof of the statement logically sound? If not, where is the flaw?

If the proof is logically sound then the statement is a true statement. Thus the whole geometry of Euclid collapses. Ask your students for their comments. If you are lucky to have really mathematically gifted students, they might be able to give you a satisfactory answer to the above paradox.