SOME ASPECTS OF EULER'S THEOREM*

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1. Warning

This article is based on a lecture at the beginning of which members of the audience were requested not to ask for clarification of points of detail during the lecture itself. The reason for this is that such requests might have spoilt the plan of the lecture. It was hoped that at the end the audience would understand why this was so.

In the case of this written version a different point has to be made, but the underlying reason is the same. It is important that the reader should read the whole article and not just parts of it. To quote some of it out of context could lead to serious misunderstanding. It is hoped that at the end the reader will understand why this is so.

2. Euler

Euler was a great mathematician. He was so great that his name is encountered frequently in mathematical literature. Terms such as 'Euler's constant', 'Euler's formula' and 'Euler's theorem' are very familiar. In fact there is some danger of confusion, for Euler invented more than one formula and proved more than one theorem.

Euler lived from 1707 to 1783. He was of Swiss nationality, but spent most of his life in Russia. His contributions to mathematics continued into old age; though blind for the last 17 years of his life, he remained a leader in the development of the subject.

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One extremely fascinating mathematical object to emerge from Euler's work is what has come to be known as the Euler characteristic of a surface, or of a more general type of space; it is often referred to as the Euler-Poincaré characteristic, in deference to the contribution at the end of the nineteenth century by the French mathematician Poincaré. One of the great theorems of mathematics is the result sometimes known as the Gauss-Bonnet theorem, which relates the curvature of a certain type of surface to its Euler characteristic. The theorem takes the form

$$\iint K dS = 2\pi\chi,$$

where K stands for the curvature and χ stands for the Euler characteristic. The significance of this theorem is that it relates curvature, something referring to how the surface behaves locally, to the characteristic, which refers to a global property: that is, a property of the surface as a whole. Even this splendid theorem is just a special case of something more general, but I shall not go into any details, for I am concerned with the theorem which produced the Euler characteristic of a surface rather than the characteristic itself. However, the words 'local' and 'global' should be remembered.

3. Polyhedra and Euler's formula

In 1750 Euler was concerned with the classification of polyhedra. I do not know precisely what his definition of polyhedron was, but I think it was similar to Legendre's definition, given in 1794.

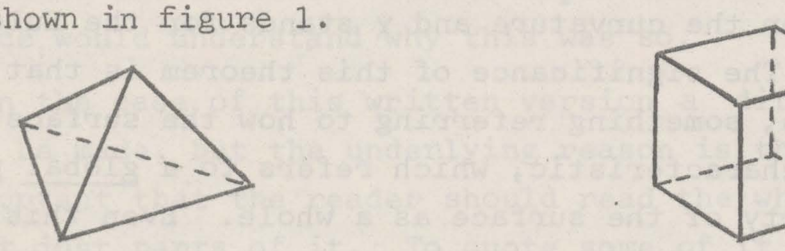
Definition 1 A polyhedron is any solid bounded by planes or plane faces.

The Greeks had evidently considered polyhedra long before Euler's time. In particular, Euclid observed that there are just five regular polyhedra, a fact which we shall consider again later. It seems rather surprising that the Greeks, and their successors up to the time of Euler, failed to put on record an important fact concerning the numbers of vertices, edges and faces of a polyhedron, although Descartes came very close to stating it.

Consider the different faces of a polyhedron. These are bounded by polygons, each having edges and vertices. We count the numbers of vertices, edges and faces for the whole polyhedron.

Notation Let V denote the number of vertices,
let E denote the number of edges and
let F denote the number of faces.

Simple examples of polyhedra are the tetrahedron and the cube, shown in figure 1.



Tetrahedron $V = 4$ $E = 6$ $F = 4$ Cube $V = 8$ $E = 12$ $F = 6$

Figure 1

Through observation, conjecture and testing, Euler arrived at the conclusion that for all polyhedra there was a relationship between V, E and F, namely

$$V - E + F = 2. \qquad (1)$$

This has come to be known as Euler's formula (see notes 1 and 2). When he first mentioned the result, he was not satisfied that he had proved it. In 1751 he did put forward a proof, but this does not seem to have been acceptable to mathematicians.

4. Euler's theorem

In 1811, some years after Euler's death, Cauchy, another great mathematician, put forward a proof which gained general acceptance amongst mathematicians. Thus the truth of equation (1) was established to their satisfaction, and could be given the status of a theorem. It can be set out tidily as follows:

for convenience the proof has been divided into three stages.

Theorem 1 (Euler's theorem) For any polyhedron, the numbers of vertices, edges and faces satisfy

$$V - E + F = 2.$$

Proof Stage 1 Imagine the polyhedron to be hollow and made of some thin flexible material. Cut out one face and stretch the remaining surface flat, without tearing or joining different portions together. We have lost a face and so aim to show that

$$V - E + F = 1$$

for what is left, namely a network of points and lines. An example is shown in figure 2; here we are treating the cube.

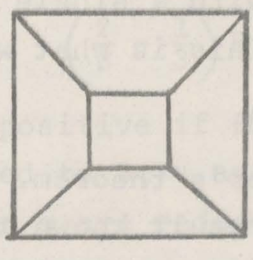


Figure 2

Stage 2 Triangulate the figure by drawing diagonals so that each portion becomes a triangle. This process increases both E and F by 1 at each stage, and so does not affect V - E + F. In figure 3, a triangulation of the network of figure 2 is shown.

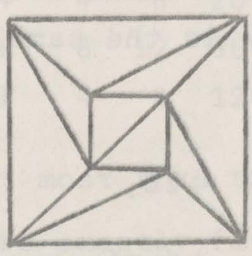


Figure 3

Stage 3 Remove the triangles one by one. This is done by either removing an edge, which results in a reduction of one face and one edge, or two edges and a vertex, which results in a reduction of one vertex, two edges and a face. Again V - E + F remains unaltered. Figure 4 illustrates one way of removing the first three triangles for our network.

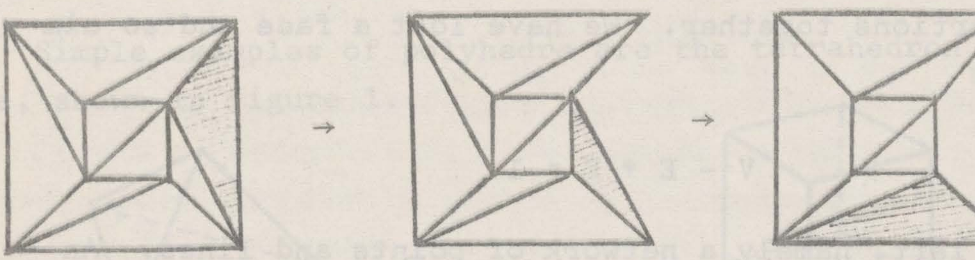


Figure 4

At the end we are left with a single triangle, for which V - E + F = 3 - 3 + 1 = 1. This is what we require.

5. The five regular polyhedra

Thus we have proved Euler's theorem. Now let us use it to prove another celebrated result known to the Greeks and already mentioned. We must first explain what is meant by a regular polyhedron.

Definition 2 A polyhedron is regular if its faces are all alike, its edges are all alike and its vertices are all alike.

Theorem 2 There are exactly five types of regular polyhedra.

Proof Stage 1 Let Π be a regular polyhedron, having V vertices, E edges and F faces. Since Π is regular, the number of edges terminating at a vertex is the same for all vertices. Let this number be α. Then

$$\alpha V = 2E, \qquad (2)$$

since each edge terminates in exactly two vertices.

Since Π is regular, the number of edges surrounding a face is the same for all faces. Let this number be β. Then

$$\beta F = 2E, \qquad (3)$$

since each edge is adjacent to exactly two faces.

By Euler's theorem,

$$V - E + F = 2$$

and so, using (2) and (3),

$$\left(\frac{2}{\alpha} + \frac{2}{\beta} - 1\right)E = 2. \qquad (4)$$

Clearly $\alpha \geqslant 3$ and $\beta \geqslant 3$, for otherwise we would not have a polyhedron. If α and β are both $\geqslant 4$, the left-hand side of (4) cannot be positive, and so (4) cannot be satisfied. Hence either $\alpha = 3$ or $\beta = 3$.

When $\alpha = 3$, the left-hand side of (4) becomes

$$\left(\frac{2}{\beta} - \frac{1}{3}\right)E$$

and again this is non-positive if $\beta \geqslant 6$. Therefore the possible values for β are reduced to 3, 4 and 5. Similarly when $\alpha = 3$, the possible values for α are reduced to 3, 4 and 5.

Given α and β we can determine E from (4) and can then obtain V and F from (2) and (3).

The above can be summarised in the following table, which gives the only possible values for α, β, V, E and F :

α	3	3	3	4	5
β	3	4	5	3	3
V	4	8	20	6	12
E	6	12	30	12	30
F	4	6	12	8	20

Hence there are at most five types of regular polyhedra.

Stage 2 There are exactly five types, because we can construct polyhedra corresponding to each of the possible pairs of values for α and β in the table. They are

(i) the tetrahedron ($\alpha = 3 \quad \beta = 3$),
(ii) the cube ($\alpha = 3 \quad \beta = 4$),
(iii) the dodecahedron ($\alpha = 3 \quad \beta = 5$),
(iv) the octahedron ($\alpha = 4 \quad \beta = 3$),
(v) the icosahedron ($\alpha = 5 \quad \beta = 3$).

This completes the proof of Theorem 2.

6. Imré Lakatos

Imré Lakatos was a mathematical philosopher, who came as a political refugee from Hungary to Britain in 1956, following the turmoil of events in the autumn of that year. He studied at Cambridge and eventually became Professor of Logic at the London School of Economics, having rapidly established himself as a leading figure in the philosophy of mathematics.

In February 1974 his career was tragically cut short by his early death. Writing an obituary notice in the Times of London, his colleague Ernest Gellner described him as one of the most brilliant thinkers and lecturers of the middle generation. He went on to say 'He had been a star member of the most important Marxist school of thought in this century and subsequently also a major contributor to the finest intellectual liberal movement of the day. The particular quality of his brilliance reflected the blending of these two traditions... . He lectured on difficult abstract subjects riddled with technicalities, the philosophy and history of mathematics and science, but he did so in a way which made it intelligible, fascinating, dramatic and above all conspicuously amusing even for non-specialists.'

What is the connection between Lakatos and Euler's theorem? The answer is that the theorem and its history provide an excellent illustration for some of the important philosophical ideas which Lakatos put forward. These ideas are concerned with the way in which modern pure mathematics is presented. The accepted system is to follow the method of 'mathematical formalism', which Lakatos questioned, for he believed that it had serious deficiencies and that a challenge to its predominance was overdue.

7. Lakatos's challenge to mathematical formalism

Most of us are familiar with this generally accepted approach to modern abstract mathematics, which exerts a considerable influence on our presentation of text-books and research papers, our teaching to undergraduates and postgraduates, and even our way of thinking. The usual procedure is to begin with the statement of the axioms, which will involve certain undefined objects, and then to continue to the basic definitions, after which come the theorems and their proofs. Further axioms and definitions may follow as the theory is built up and becomes more complicated. Perfection in the proofs of theorems, within the limits prescribed by certain (usually unwritten) conventions, is essential. If a theorem is newly presented and the argument is seen to be wrong, this reflects on both the theorem and its author.

Explanations to account for the choice of axioms, to reveal the origin of the definitions and to put the case for developing the theorems, are often inadequate or totally absent. Some say that such explanations are irrelevant, because they are essentially outside the perfected mathematical theory. What is of interest is the theory itself as a piece of mathematics. To understand it and to assess its quality requires mathematical maturity and an appreciation of mathematical beauty. As far as the axioms, definitions and theorems are concerned, you can take them or leave them; but if you leave them, the chances are that you are not a real mathematician.

This is the approach of mathematical formalism, which emerged strongly from the work of such mathematicians as Hilbert, and which now exerts a powerful hold on pure mathematicians throughout the world.

It is of interest that students of mathematics rarely question the method of presentation demanded by mathematical formalism (see note 3). In some ways this is strange, because mathematics presented in this way is an authoritarian subject and modern students are noted for their reluctance to accept authority. Perhaps one reason is that this form of authoritarianism is easy to accept, because the rules are clear and the rewards are attractive. Learning how to play the game is not too hard; axiom-manoeuvring is a good deal easier than problem-solving.

Lakatos disagreed with the approach of mathematical formalism, the 'deductivist' approach, as he called it. He championed the 'heuristic' approach. The word 'heuristic' means 'serving to discover' and the heuristic method is essentially that of finding things out for oneself. He believed that many parts of mathematics could only be understood through a study of their history. Mathematics has progressed not by what the perfected theories of the formalists would apparently have us believe, namely 'a monotonous increase of indubitably established theorems', but through the incessant improvement of guesses by speculation, criticism, argument and debate.

Thus in using the heuristic approach to a mathematical theorem we note certain facts about the objects of interest, we make conjectures, we test them, experimenting with proofs; we find flaws, perhaps counterexamples; we go back, modify, and try to perfect. This is the 'method of proofs and refutations', and Lakatos believed that mathematical papers should be presented from this point of view rather than in the cold, austere fashion which is now generally regarded as appropriate. One result would be that papers would become much longer, but this would be offset by the disappearance of some, the publication of which would be seen to be unjustifiable because their lack of significance would become apparent; mathematical formalism can obscure the fact that a theorem is of no importance.

In practice, mathematicians frequently do not build their theories in the way suggested by mathematical formalism, but use to a greater or lesser degree the heuristic approach, at least in the early stages. But formalism takes over. It can lead, as already stated, to the inclusion of results of no significance, and from the point of view of teaching it suppresses an important part of the truth.

8. Proofs and refutations

Lakatos illustrated his ideas in papers (note 4) published in the British Journal for the Philosophy of Science in 1964. Subsequently these papers were published, along with further material, in book form (note 5).

Euler's theorem is used as the main illustration of the ideas. The material is cast in the form of a dialogue in a classroom. The developments in this dialogue follow the actual historical developments concerning the theorem, the relevant references being given and liberally commented on in the footnotes.

We have a teacher and a number of pupils, the latter identified by Greek letters. Clearly these pupils are highly intelligent and not at all willing to submit to mathematical authoritarianism. They are capable of developing the theory as mathematicians over the years developed it. The mathematics which emerges is fascinating, but it is the manner in which it emerges that captures the imagination.

The only way to appreciate Lakatos's work in full is to read his book for yourself. I shall just describe the first few pages in the light of what I have already set down about Euler's theorem.

9. The proof of Euler's theorem

The class begins with the conjecture V - E + F = 2 for polyhedra, which they have arrived at as a result of testing, observing and guessing. The teacher presents a proof, which he describes as a 'thought-experiment'. It is essentially that due to Cauchy, which was accepted by most mathematicians of his day as convincing; it is the proof given in §4 above. He ends by saying 'thus we have proved our conjecture'.

One pupil seems to accept it, but not everyone is satisfied. Pupils α, β, γ attack the three stages of the proof in turn.

α: 'I see that this experiment can be performed for a cube or tetrahedron, but how am I to know that it can be performed for any polyhedron? Is it true that any polyhedron, after having a face removed, can be stretched flat? I am dubious about your first step.'

β: 'Are you sure that in triangulating the map one will always get a new face for any new edge? I am dubious about your second step.'

γ: 'Are you sure that there are only two alternatives - the disappearance of one edge or else of two edges and a vertex - when one drops the triangles one by one? Are you even sure that one is left with a single triangle at the end of this process? I am dubious about your third step.'

The teacher agrees that he is not sure. He suggests that the class should look at the proof carefully, regarding it as being decomposed into three separate parts to give three lemmas and then to consider the possibility of counterexamples.

Pupil γ produces a counterexample (see note 6) to Lemma 3. If we start by removing a triangle from the <u>inside</u> of the network then the first step does <u>not</u> change the number of edges and vertices but a face is lost. Only in the case of the tetrahedron does this fail to give a counterexample; for the tetrahedron there are no 'inside' triangles.

The teacher points out that this refutes the lemma but not the theorem, which still holds for the cube even though in this case γ's argument produces a counterexample to the lemma. Thus the counterexample is a <u>local</u> one, for it refutes the argument but not the theorem. A <u>global</u> counterexample would be one which refuted the theorem.

The teacher's answer to the difficulty is to replace Lemma 3 by a new one. In this, he insists on removing only <u>boundary</u> triangles at each stage.

But this, too, is wrong, for γ can again produce a counterexample: he proposes the process indicated in figure 5,

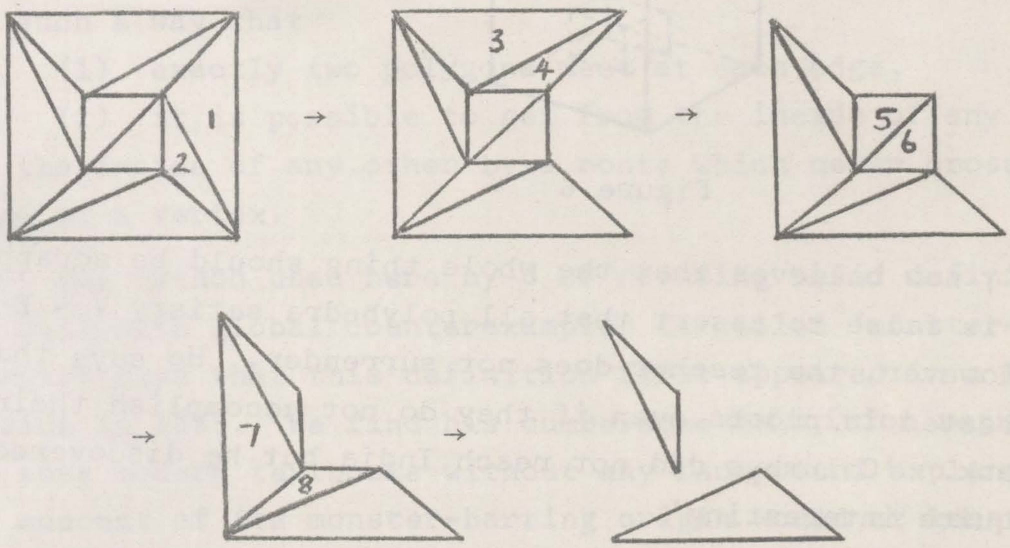


Figure 5

which leaves two separate triangles and so leads to V - E + F = 2, rather than V - E + F = 1, which is what is required in this part of the proof.

The teacher might have answered this one by saying that in the new version of Lemma 3 it should be stated that the triangles can be removed in such a way that their removal does not disconnect the network. However he believes that it would be better to adopt the following version: 'the triangles in the network can be numbered so that in removing them in the right order V - E + F will not alter until we reach the last triangle'.

This causes some criticism in the class. It is observed that the original Lemma 3 seemed to be trivially true, but the new version does not look plausible enough; how can we believe that it can escape refutation?

But next there is a dramatic development, for pupil α produces a global counterexample. This consists of a pair of nested cubes, as shown in figure 6. For this, V - E + F = 4 and so the theorem is false.

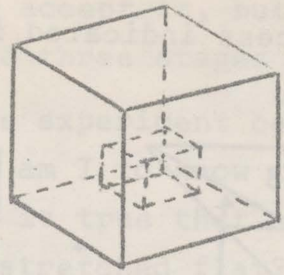


Figure 6

Pupil γ now believes that the whole thing should be scrapped. Clearly it is false to assert that all polyhedra satisfy V - E + F = 2. However, the teacher does not surrender. He says that he is interested in proofs even if they do not accomplish their intended task. 'Columbus did not reach India but he discovered something quite interesting'.

Pupil δ enters into an argument with pupil α. He says that the counterexample produced by α is fake criticism, for this is not a polyhedron at all; it is a monster , a pathological case. He counters α's claim that nevertheless it satisfies the definition of polyhedron (see Definition 1 in §3 above) by stating that the definition itself is at fault. This should be abandoned, and replaced by the following.

Definition 3 A polyhedron is a surface consisting of a system of polygons.

The answer to this from α consists of the construction of two new global counterexamples, the 'twin-tetrahedra', shown in figure 7.

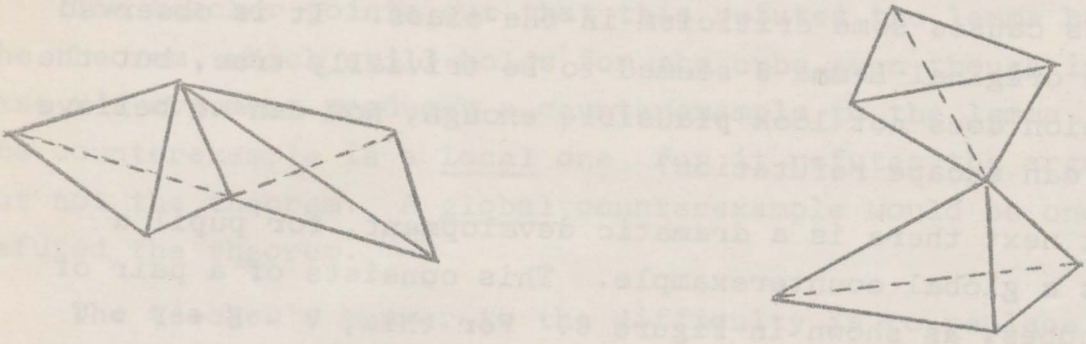


Figure 7

For each of these, V - E + F = 3.

Pupil δ, whilst 'admiring α's perverted imagination' adjusts the definition of polyhedron again:

Definition 4 A polyhedron is a system of polygons arranged in such a way that

(1) exactly two polygons meet at each edge,

(2) it is possible to get from the inside of any polygon to the inside of any other by a route which never crosses any edge at a vertex.

The method used here by δ of revising basic definitions to deal with global counterexamples is called 'monster-barring'. Lakatos says that this definition first appeared in a book by Möbius in 1865. We find his cumbersome definition reproduced in some modern textbooks without any background explanation; an account of its monster-barring origin would at least explain why the definition is given in this way. Definitions are often influenced by what has happened in formulating proofs, but the story usually remains untold.

Pupil α now admires δ's perverted ingenuity in 'inventing one definition after another as barricades against the falsification of your pet ideas. Why don't you just define a polyhedron as a system of polygons for which the equation $V - E + F = 2$ holds, and this perfect definition would settle the dispute for ever?'

10. Conclusion

Is that the end? No! The dialogue has scarcely begun. The techniques of the method of proofs and refutations have not yet been analysed in full and many significant historical developments have not yet been covered. There is much more to be said; many thought- experiments, criticisms, counter-examples, arguments, counter-arguments and revisions of ideas are to come.

I hope that I have whetted your appetites. Go back to the beginning of the article, with a critical eye, and see how much you can find to argue about. Can you improve further the definition of polyhedron, the statement of the theorem and its proof until you can be sure that you have a sensible and interesting (possibly useful) theorem, with no global counterexamples to the result itself and no local counterexamples to the proof? Does all this satisfy the criteria required by mathematical formalism? If so, should you suppress all that you did to perfect your theory?

Maybe this is asking too much, but at least it is worth your while to read how Lakatos unfolds the story of the development of Euler's theorem, to think about its message and to consider whether it is in the spirit of Lakatos that mathematics should be presented to the world.

Notes and References

1. The expression V - E + F is the Euler characteristic referred to in §2. Thus Euler's formula implies that the characteristic of a polyhedron is 2.

2. It may be of interest to note that there is a formal resemblance between Euler's formula and the phase rule in Chemistry. This says that in a chemical reaction if P is the number of phases, C the number of components and F the number of degrees of freedom, then

$$P - C + F = 2.$$

3. A question which <u>is</u> popular with students is a different one: what use is this theory? This requires a separate article.

4. 'Proofs and Refutations', British Journal for the Philosophy of Science, XIV, 1964; part I 1-25, part II 120-139, part III 221-245 and part IV 296-342.

5. *Proofs and Refutations*, Cambridge University Press, 1976

6. It is hoped that the significance of §1 is now becoming clear.