NOTES ON MATHEMATICIANS

7. Hermann Weyl (1885 - 1955)

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Since the outburst of intense activities in mathematics and physics during the first two decades of this century, with the concomitant expansion of perspectives in human knowledge, answers to old problems have only stirred up fresh queries which demanded more answers to new problems. Specialized efforts would be required to provide these answers which, in turn, opened up new regions for scientific endeavour. As each explorer maps out and settles confortably into his own niche of territory, it would seem well-nigh impossible for a mortal mind to pick its way during one human lifetime from one labyrinth of knowledge to another, let alone to slay the Minotaur of each labyrinth.

Throughout the ages, the great mathematical minds of each epoch have pondered over a wide spectrum of knowledge - Archimedes [1], Isaac Newton [2], Joseph Louis Lagrange [3], Leonhard Euler [4], Carl Friedrich Gauss [5], Henri Poincaré [6], David Hilbert [7], Hermann Weyl, John von Neumann [8], to name only a few. The two great upholders of this tradition in the nineteenth century were Poincaré and Hilbert. The latter created the Göttingen school of mathematics which influenced and directed the development of much of modern mathematics and, to some extent, modern physics. Hilbert was Weyl's mentor. In characteristically Hilbertian fashion, Weyl devoted different stages of his life to the intensive study of different subjects in a concerted attempt to inject new ideas, effect fundamental changes and discover underlying connections.

As the theoretical physicist Freeman J. Dyson [9] writes in the obituary published in the scientific journal *Nature* : "Among all the mathematicians who began their working lives in the twentieth century, Hermann Weyl was the one who made major contributions in the greatest number of different fields. He alone could stand comparison with the last great universal mathematicians of the nineteenth century, Hilbert and Poincaré. So long as he was alive, he embodied a living contact between the main lines of advance in pure mathematics and in theoretical physics. Now he is dead, the contact is broken, and our hopes of comprehending the physical universe by a direct use of creative mathematical imagination are for the time being ended." [10]

The making of a mathematician Weyl was born on 9 November 1885 in the small town of Elmshorn near Hamburg in Germany. His parents were Ludwig and Anna Weyl, and he spent his school days in Altona. The director of his gymnasium (a school that prepares its students for the university) happened to be a cousin of Hilbert, then a professor at Göttingen. So, in 1903, he went to further his studies at the University of Göttingen. Weyl appeared as an eighteen-year-old country lad, shy and inarticulate but confident of his own abilities. As he has written, "In the fullness of my innocence and ignorance, I made bold to take the course Hilbert had announced for that term, on the notion of number and the quadrature of the circle. Most of it went straight over my head. But the doors of a new world swung open for me, and I had not sat long at Hilbert's feet before the resolution formed itself in my young heart that I must by all means read and study whatever this man had written." [11]

Except for one year at the University of Munich, Weyl's mathematical education was completed at Göttingen. During this period, the two bright stars that lit up the mathematical firmament of this scientific centre were Hilbert and Hermann Minkowski [12], both of them at the peak of their careers. The aging Felix Klein [13], the grand old man of German mathematics who himself embodied a living legend, contented himself in directing the development of Göttingen as the centre of the scientific world. The universities that subsequently sprang up in America together with their scientific-technological complexes were patterned after Göttingen.

At the end of the first year, Weyl returned home with a copy of Hilbert's *Zahlbericht (Die Theorie der algebraischen Zahlkörper)*, the 1896 monumental report that laid the foundations of the modern theory of algebraic numbers. The following months of the summer vacation which Weyl, without any previous knowledge of elementary number theory or Galois theory, spent in going through the report, were, in his own words, "the happiest months of my life, whose shine, across years burdened with our common share of doubt and failure, still comforts my soul." [11]

It was indeed his great fortune that Weyl spent his formative years and came of age as a mathematician during the golden age of the history of Göttingen. He obtained his doctorate in 1908 and became a Privatdozent [14] in 1910. One marvels at the long list of impressive names, immortalized in the roll-call of pioneers and creators of mathematical and scientific thought, that made their debut as budding talents in Göttingen. In the words of Weyl, "A university such as Göttingen, in the halcyon days before 1914, was particularly favourable for the development of a living scientific school. Once a band of disciples had gathered around Hilbert, intent upon research and little worried by the chore of teaching, it was but natural that in joint competitive aspiration of related aims each should stimulate the other; there was no need that everything come from the master." [15]

Some of Weyl's contemporaries in Göttingen were Max Born [16], Richard Courant [17], Harald Bohr [18], Erich Hecke [19] and George Pólya [20]. There were cliques and in-groups that ruled over the social life of the mathematical world of Göttingen. But Weyl, even after becoming a Privatdozent, was still too shy for such in-groups. It therefore came as a shock when he won the hand of a lady "whose charms were such that when her father threatened to withdraw her from the University a petition begging him to reconsider was signed even by professors" [21]. In September 1913, he married Helene Joseph, the daughter of a doctor and the translator into German of many Spanish writings. Out of this marriage two sons were born, one of whom, Fritz Joachim, became a mathematician.

Weyl's doctoral dissertation dealt with singular integral equations. The theory of integral equations was first initiated by Ivar Fredholm [22] in 1903, and, in the hands of Hilbert, bore fruit to reveal the underlying relationship between integral equations and linear algebra, leading to a general theory of so-called Hilbert spaces. As a natural continuation of his dissertation, Weyl applied the theory of integral equations to singular eigenvalue problems of ordinary differential equations in a long series of papers published during 1908 - 1915.

Appearing in parallel with the above series was his work on the asymptotic distribution of natural frequencies of oscillating continua such as membranes, elastic bodies and electromagnetic waves. The result had been conjectured earlier by physicists and the discovery of its mathematical proof was decisive to Weyl. He recalled how the kerosene lamp began to fume as he worked feverishly on the solution, and when he had finished, there was a thick layer of soot on his papers, hands and face. [23]

Integral equations occur frequently in physics, and the classical problems of the oscillations of continua and of potential theory would have been intractable without a proper theory of these equations. So it was from the beginning that Weyl's first researches in analysis, a branch of pure mathematics, had brought him face to face with problems of the natural sciences — the beginning too of a life-long interest in applying mathematics to the sciences.

As a Privatdozent in Göttingen, he gave a course on the classical theory of algebraic and analytic functions on Riemann surfaces. Out of this appeared in 1913 his first book, *Die Idee der Riemannschen Flache* (*The concept of a Riemann surface*), which was "to exert a profound influence on the mathematical thought of his age". [23] For the first time, it laid the rigorous foundations of "geometrical" function theory and at the same time linked together analysis, geometry and topology (it was only a few years before, in 1910 - 1911, that the first basic topological notions were given by L. E. J. Brouwer [24]). Forty-two years later, in 1955, Weyl published a new edition of the book and recast it in the by now familiar language of topology.

Controversy amidst tranquillity The years 1913 - 1930 formed a period of comparative tranquillity in Weyl's life. He went to Zürich in 1913 at the age of twenty-eight to take up a professorship of mathematics at the Swiss Federal Institute of Technology (Eidgenossische Technische Hochschule). The First World War exploded, and he was enlisted into the Swiss army and served for one year as a private with a garrison at Saarbrücken. At the end of his service, he returned to lecture at Zürich. He was happily married, with his wife sharing "to the full his taste for philosophy and for imaginative and poetical literature" [25]. The war came and passed by Switzerland, whose neutrality in the conflict ensured that its peace and calm was not breached while the rest of Europe was tearing itself apart.

His stature in mathematics and physics grew. He took an active interest in the mathematical and scientific revolutions of that era. The 1915 theory of general relativity of Albert Einstein [26], who did a two-year spell from 1912 to 1914 at the Federal Institute, attracted and stimulated him into the search for a "unified field theory", an extension to cover gravitation and electromagnetism. At the same time, he was swept as a partisan into the deep currents of controversy that lashed at the foundations of mathematics. He found himself in the camp of the "Intuitionists" led by Brouwer, the arch-enemy of Hilbert in mathematics.

At the height of his powers, he published papers in number theory, differential geometry, harmonic analysis and its applications to quantum mechanics. He was interested in practically all fields of creative human endeavour, including art, music, philosophy and literature. His creative spirit sought to express itself wherever it could. Even in mathematics, he thought of himself as an artist. As he has said in retrospect many years later in Zürich, "Expression and shape are almost more to me than knowledge itself. But I believe that, leaving aside my own peculiar nature, there is in mathematics itself, in contrast to the experimental disciplines, a character which is nearer to that of free creative art. For this reason the modern scientific urge to found Institutes of Science is not so good for mathematics, where the relationship between teacher and pupil should be milder and looser. In the fine arts we do not normally seek to impose the systematic training of pupils upon creative artists." [25]

He was still in his thirties when his fame and reputation emanated from Zürich. There were numerous offers of chairs by foreign universities. In 1922, Klein and Hilbert asked him to return to the fold of Göttingen. Because of his respect for them, Weyl pondered over the decision for a long time. He was so worried that he walked his wife Hella round and round the block of their home until nearly midnight before he decided to accept the offer. He then rushed off to send a telegram of acceptance, but returned home after he had telegraphed to reject the offer instead. He was not prepared "to exchange the tranquillity of life in Zürich for the uncertainties of post-war Germany". [21]

In the summer of 1917, Weyl gave a course of lectures on the theory of general relativity and published it in 1918 as *Raum-Zeit-Materie (Space, time, matter)* which went through five editions in five years to become the 1923 classic. Once his interest was kindled, he set out to generalize Einstein's theory and produced in 1918 a "unified field theory" — the first of many such theories, in an as yet unfinished quest pioneered by Weyl, Arthur Eddington [27], T. Kaluza [28] and Einstein himself. By introducing the possibility of a change of size in the geometry of space-time, Weyl was able to interpret electromagnetism as an aspect of this geometry, just as Einstein interpreted gravitation as curvature when he introduced a change of direction. Mathematically and aesthetically satisfying though it was, Weyl's theory had to be abandoned because of an objection raised by Einstein — that the size of an object would then depend on its history, with the result that chemically pure elements would not exist and spectral smears would be observed instead of spectral lines.

Weyl's attempt at unified field theory had a greater influence on differential geometry. In his 1929 papers on gravitation and the electron, he applied Einstein's notion (of attaching a local set of axes to each point of space-time) to quantum mechanics and interpreted it in terms of spin. The describability of certain particles called "fermions" by means of algebraic entities called spinors, first pointed out by Weyl in these papers, is now vindicated by the modern theory of neutrinos. The wave equations obtained by Weyl are, strangely enough, not invariant under spatial reflection ("non-conservation of parity" [29]), and one of them turns out to be a suitable description of the so-called "weak interactions" (the forces occurring in nature being the strong or nuclear interaction, the electromagnetic interaction, the weak interaction and the gravitational interaction in decreasing order of strength). [30]

An off-shoot of Weyl's geometrical investigations in general relativity was his subsequent interest in the so-called "space-problem" of getting to the "root" of the structure of a general metric space. Its solution was published in 1923 as a book, *Mathematische Analyse des Raumproblems (Mathematical analysis of the space-problem)*. This led naturally to the general problem of the representation of continuous groups. In his three classical papers of 1925 - 1926, he made the first significant contributions to the global study of Lie groups and blazed a trail for modern research. One of the high points of his investigations is the well-known "Peter-Weyl theorem" of 1927.

At the same time, quantum theory was taking long strides following the initial breakthroughs of the previous decade. An understanding of the microcospic world of the atom gradually emerged with the profound contributions of physicists like Born, Werner Heisenberg [31], Wolfgang Pauli [32], P.A.M. Dirac [33], Eugene Wigner [34] and Erwin Schrödinger [35]. With his discerning eye for relationships, Weyl saw that his newly developed theory of continuous groups could play a vital role in the mathematical formulation of quantum theory. Thus appeared in 1928 his classical work *Gruppentheorie und Quantenmechanik (The theory of groups and quantum mechanics)*. Its influence may be summed up in these words of Dyson : "By bringing group theory into quantum mechanics he led the way to our modern style of thinking in physics. Today the instinctive reaction of every theoretical physicist, confronted with an unexplained regularity in the behaviour of elementary particles, is to postulate an underlying symmetry-group." [10]

Another foundation-shaking event in the first decade of this century was Brouwer's revolt against the absolute validity of classical (Aristotelian) logic. One of the two "laws" of the classical logic, the so-called "law of the excluded middle", states that a given statement is either true or false. Classical mathematics, or mathematics up to the late nineteenth century, has always assumed the universal validty of this law when applied to a given mathematical system (in particular, number theory). The existence of mathematical objects is deemed to be established once their non-existence could be shown to lead to a contradiction within the system even though the proof itself gives no inkling as to what the objects are. Earlier, L. Kronecker [36] had objected strongly to the use of the "actually infinite" in mathematics and would only accept constructive definitions and proofs. He mercilessly criticized the works of Richard Dedekind [37] in analysis and of Georg Cantor [38] in set theory. This seemingly iconoclastic attitude found little support or sympathy from the rest of the mathematical community. However, in his 1907 thesis on the limitations of the law of the excluded middle, Brouwer reopened the old wounds of Kronecker's attack and started a systematic programme to rebuild mathematics on the "intuitionistic" philosophy that all mathematics may be based on the integers and developed by means of "intuitively clear" constructive methods. [39]

These ideas propagated to Zürich and found in the young Weyl a receptive mind. "Brouwer opened our eyes and made us see how far classical mathematics, nourished by a belief in the 'absolute' that transcends all human possibilities of realization, goes beyond such statements as can claim real meaning and truth founded on evidence." [40] With youthful zest and missionary zeal, Weyl championed the cause of the Instuitionist school, much to the dismay of his old teacher, Hilbert. Fearing that much of what was dear and valuable to him would be jettisoned from the bulk of mathematics, Hilbert countered with his own "formalist" programme of establishing the consistency of mathematics by reducing mathematics to a formal game of meaningless symbols — an approach which was foreign and repugnant to Weyl.

A monograph, *Das Kontinuum (The continuum)* appeared in 1918 in which Weyl laid down some of his own ideas, and an exposition of his stand appeared in his book *Philosophie der Mathematik und Naturwissenschaft* (translated and expanded in 1949 as *Philosophy of mathematics and natural science*), which was published in the 1927 *Handbuch der Philosophie*. Although his contributions to the foundations of mathematics were not as incisive as his other efforts, his writings gained for the intuitionists a wider audience and "turned the revolutionary doctrine of his time into the orthodoxy of today." [25]

Among some of his work done during the Zürich period, but which do not fit into any of the above general themes, are his papers on "uniform distribution modulo 1", convex surfaces and almost periodic functions. A sequence of real numbers $a_1, a_2, \ldots$ is said to be uniformly distributed modulo 1 if, for each positive integer n and each positive real number c, A(n,c)/n tends to c as n tends to infinity, where A(n,c) is the number of those of $a_1, \ldots, a_n$ which have fractional parts between 0 and c. In his classical papers of 1916, Weyl gave a necessary and sufficient condition for a sequence to be uniformly distributed modulo 1, namely that, for each non-zero integer h, $\frac{1}{n}\sum_{k=1}^{n} e^{2\pi i h a_k}$ tends to zero as h tends to infinity. He also developed a method of estimating certain exponential sums essential to the study of Waring's Problem [41] and of the estimation of the Riemann zeta-function [42]. This work came close to the theory of almost periodic functions, whose foundations were laid by Bohr in 1924 and on which Weyl later published some results.

Earlier in Göttingen, Weyl had collaborated with Hilbert in the publication of Minkowski's collected works and was stimulated by Minkowski's theory of convex bodies. On his return to Zürich, he published in 1916 and 1917 some work in this field, but his interest by then had shifted to general relativity.

Uncertainty The Zürich period 1913 - 1930 was punctuated by a short stint (1928 - 1929) as Jones research professor in mathematical physics at Princeton University. His return to Zürich was confronted by an offer from Göttingen to succeed Hilbert at the Mathematical Institute, whose post-war development had progressed steadily under the efforts of Courant until its abrupt cessation by the Nazi policy of preserving the Aryan "purity" of German universities. Weyl had refused an earlier opportunity but now, in his forties, he could not resist the call of the great Göttingen tradition that had seemingly survived intact from the privations of the early post-war period. After some hesitation, he finally made the crucial decision of accepting the offer. And in the spring of 1930, he arrived in Göttingen with optimism and expectation. What he had hoped for turned out to be a brief sojourn of tension and uncertainty. "The three years that followed were the most painful that Hella and I have known." [23]

In 1932, German mathematicians celebrated the seventieth birthday of Hilbert. Weyl wrote a birthday greeting in *Die Naturwissenschaften*. Friends, former colleagues and students converged at Göttingen for the occasion which would be the last supper of Göttingen mathematicians. In the same year, the National Socialist Party came to power and a systematic campaign was soon set in motion to purge all Jews from the intellectual, political and economic spheres of German life. The axe fell on the Mathematical Institute. Those mathematicians who were classified as Jews were dismissed and forced to migrate. Those who were not dismissed but who understood the meaning of this also left Germany in disgust.

Meanwhile, Weyl took over as the head of the Mathematical Institute and tried hard to salvage the situation. By late summer of 1933, it was evident that nothing could be salvaged. Weyl and his family were back in Switzerland on vacation. Friends wrote to him from America pleading with him to leave Germany before it was too late. The newly created Institute for Advanced Study at Princeton offered him a position. Finally, after an agonizing period of hesitation, he took Einstein's advice to join him at the Institute.

Princeton and thereafter. From Zürich, Weyl sent to Göttingen his letter of resignation and arrived at Princeton towards the end of 1933 to take up a professorship of mathematics at the Institute for Advanced Study. He held the position until his retirement in 1952. The initial years at the Institute were years of painful adjustment to an unfamiliar enviroment and a foreign tongue. But he responded to the challenge and overcame it. His newly gained happiness was for a while shattered by the death of his wife Hella in 1948. This was a cruel blow to him and his usual self was only restored by a second marriage in 1950 to Ellen Baer Lohnstein of Zürich. He would henceforth divide his time between Princeton and Zürich. It seemed that [43], as a result of living in Switzerland for a time, he violated the permissibility for a naturalized American citizen to stay abroad and retain his citizenship. His loss of American citizenship by negligence caused an uproar among American mathematicians. Efforts were made to restore his citizenship. But on 8 December 1955, one month after his seventieth birthday celebration, Weyl went out to post a letter in Zürich and suffered a fatal heart attack.

When Weyl migrated to America, he was in his late forties, and he knew that the bulk of his achievements lay behind him. But he was not content to rest on his laurels. He could yet help to build at the Institute a great centre for world mathematics. Most of his time was now devoted to expanding on and simplifying the intricacies of the great themes of his mathematical work. Thus in his book *The classical groups, their invariants and representations* published in 1939, he gave a purely algebraic treatment of his well-known results of 1926 and, in addition, set out to relate these results with the theory of invariants whose central problems had been "finished once and for all" (in his own words) by Hilbert. With this new synthesis, "one can ask whether Weyl had not, in fact, delivered its *coup de grace*". [23]

He lectured and wrote in English with the fine strokes of a master. In 1952 a book *Symmetry* was published from the lectures he had given at Princeton University. It manifests, not without awe-inspiring feelings, the depth and breadth of his Promethean universality. Up to his final years, he displayed a keen interest in and sharp awareness of the achievements of the modern generation of young mathematicians. This is exemplified by his address on the occasion of the presentation of the Fields Medal Award [44] at the 1954 International Congress of Mathematicians at Amsterdam. His own growth and development as a mathematician had run parallel to the evolution of modern mathematics. "But when Weyl was asked by a publisher to write a history of mathematics in the twentieth century he turned it down because he felt that no one person could do it." [43] A sobering thought at the present stage of division and proliferation of mathematical knowledge.

When Hilbert died, only Weyl could have written by himself a comprehensive survey [11] of Hilbert's work. But when an appraisal of Weyl's accomplishments had to be written after Weyl's death, it was done through the combined efforts of five mathematicians (see [25]). Another appraisal [23] was written by two of the world's leading mathematicians. In the latter, Weyl's papers are classified under eleven broad subject headings: analysis, geometry, invariants and Lie groups, relativity, quantum theory, theory of algebras, geometry of numbers, logic, philosophy, history and biographies, and others. The papers on analysis are distributed further into twelve specialized topics; geometry into four; invariants and Lie groups into three.

In the obituary [15], Weyl writes of Hilbert: "The methodical unity of mathematics was for him a matter of belief and experience. It appeared to him essential that — in the face of the manifold interrelations and for the sake of the fertility of research — the productive mathematician should make himself at home in all fields ... . A characteristic feature of Hilbert's method is a peculiarly direct attack on problems, unfettered by any algorithms; he always goes back to the questions in their original simplicity." Perhaps these words could well have been written of Weyl himself.

Mathematics for him is not justified by its success or usefulness. The *raison d'etre* of mathematics lies in the very heart of human existence itself. "I believe that mathematizing, like music, is a creative ability deeply grounded in man's nature. Not as an isolated technical accomplishment, but only as part of human existence in its totality can it find its justification." [45] The ultimate criterion of his own work is beauty. "My work always tried to unite the true with the beautiful; but when I had to choose one or the other, I usually chose the beautiful." [10]

Notes and references

1. Archimedes (c.287B.C. - 212B.C.), Greek mathematician, physicist and engineer; probably studied with successors of Euclid (c.300B.C.) at Alexandria; greatest mathematical genius of antiquity; contributed to arithmetic, geometry, hydrostatics, mechanics; developed proofs by methods of exhaustion and reduction to absurdity.

2. Isaac Newton (1642 - 1727), English mathematician, physicist and astronomer; invented differential and integral calculus; founded theoretical mechanics; discovered law of gravitation.

3. Joseph Louis Lagrange (1736 - 1813), French mathematician and mathematical physicist; contributed to number theory, calculus of variations, partial differential equations, hydrodynamics, celestial mechanics.

4. Leonhard Euler (1707 - 1783), Swiss mathematician and physicist; studied under Johann Bernoulli (1667 - 1748) at Basle (Switzerland); spent most of his time at St. Petersburg (now Leningrad, USSR); most prolific mathematician in history; contributed to analysis, geometry, number theory, celestial mechanics, hydrodynamics, optics, accoustics, navigation.

5. Carl Freiderich Gauss (1777 - 1855) German mathematician, astronomer and physicist; established the tradition of Göttingen; exerted a universal influence on mathematics and physics through his work on number theory, analysis, geometry, celestial mechanics, geodesy, telegraphy, electromagnetism. See C. T. Chong, "Notes on mathematicians 1. Carl Friedrich Gauss", This *Medley* Vol.3, No.1 (1975), 6 - 10.

6. Henri Poincaré (1854 - 1912), French mathematician and physicist; studied at École Polytechnique École Supérieur des Mines; worked at Caen (France), Paris; contributed to analysis, celestial mechanics, number theory, geometry, fluid mechanics, electromagnetism, philosophy of science. See C. T. Chong, "Notes on mathematicians 4. Henri Poincaré", This *Medley* Vol.4,

No.1 (1976), 13 - 34.

7. David Hilbert (1861 - 1943), German mathematician and mathematical physicist; built up the school of Göttingen; contributed to algebra, number theory, geometry, analysis, foundations of mathematics, mathematical methods of physics. See C. T. Chong, "Notes on mathematicians 6. David Hilbert", This *Medley* Vol.4, No.3 (1976), 134 - 159.

8. John von Neumann (1903 - 1957), Hungarian-born mathematician and theoretical physicist; studied at Budapest, Berlin, Zürich, Göttingen; worked at Berlin, Hamburg, Institute for Advanced Study (USA); contributed to set theory, algebra, analysis, game theory, computer science, mathematical physics. See Y. K. Leong, "Notes on mathematicians 3. John von Neumann", This *Medley* Vol.3, No.3 (1975), 90 - 106.

9. Freeman John Dyson (1923 -), British-born American theoretical physicist; studied at Cambridge University; worked at Cambridge, Cornell, Princeton; professor at Institute for Advanced Study (USA); contributed to quantum electrodynamics and theory of electromagnetic radiation.

10. Freeman J. Dyson, "Obituary", *Nature* 177 (1956), 457 - 458.

11. Hermann Weyl, "David Hilbert and his mathematical work", *Bulletin of the American Mathematical Society* 50 (1944), 612 - 654. Reprinted in *Gesammelte Abhandlungen*, Band IV, Springer, Berlin, 1968.

12. Hermann Minkowski (1864 - 1909), Russian-born German mathematician and theoretical physicist; studied at Berlin, Königsberg (now Kaliningrad, U.S.S.R.); worked at Königsberg, Zürich, Federal Institute of Technology (Switzerland), Göttingen; contributed to geometry of numbers, algebra, general relativity, electrodynamics.

13. Felix Klein (1849 - 1925), German mathematician; studied at Bonn, Leipzig, Göttingen, Vienna; worked at Bonn, Göttingen, Erlangen (Germany); contributed to geometry, analysis, differential equations.

14. In the pre-war German academic system, a doctor of philosophy was required to produce another original piece of work for the so-called "Habilitation". He would then be awarded the title of Privatdozent and the privilege to lecture in the university without pay. He could, however, collect fees from students attending his lectures. Upon recognition of his work and abilities, he would receive the salaried post of an Extraordinarius (assistant professor). The final goal would be an Ordinariat (professor).

15. Hermann Weyl, "Obituary : David Hilbert 1862 - 1943", *Obituary Notices of Fellows of the Royal Society* 4 (1944), 547 - 553; *American Philosophical Society Year Book* (1944), 387 - 395. Reprinted in *Gesammelte Abhandlungen*, Band IV.

16. Max Born (1882 -), German theoretical physicist; Nobel laureate (1954); studied at Breslau (Poland), Heidelberg, Zürich, Göttingen; worked at Göttingen, Berlin, Frankfurt, Edinburgh; contributed to quantum theory and general relativity.

17. Richard Courant (1888 - 1972), Polish-born American mathematician; studied at Breslau, Zürich, Göttingen; worked at Göttingen, Münster (Germany), Cambridge, New York University; director of Courant Institute of Mathematics (New York); contributed to applied mathematics, analysis and its applications to physics.

18. Harald Bohr (1887 - 1951), Danish mathematician; brother of physicist Niels Bohr (1885 - 1962); studied at Copenhagen and Göttingen; worked at Copenhagen and College of Technology (Denmark); contributed to analysis and theory of almost periodic functions.

19. Erich Hecke (1887 - 1947), German mathematician; studied at Göttingen; worked at Basel (Switzerland), Göttingen, Hamburg; contributed to number theory (particularly theory of algebraic numbers).

20 George Pólya (1887 -), Hungarian-born American mathematician; studied at Budapest, Göttingen, Paris; worked at

Federal Institute of Technology (Switzerland), Brown, Stanford; contributed to analysis, number theory, probability, applied mathematics, mathematical pedagogy.

21. Constance Reid, *Hilbert*, Allen & Unwin, Springer, London, Berlin, 1970.

22. Erik Ivar Fredholm (1866 - 1927), Swedish mathematician; studied at Stockholm Polytechnic, Uppsala, Stockholm; worked at Stockholm; contributed to theory of integral equations.

23. C. Chevalley and A. Weil, "Hermann Weyl (1885 - 1955)" (in French), *L'Enseignement Mathématique*, tome III, fasc. 3 (1957).

24. L.E.J. Brouwer (1881 - 1966), Dutch mathematician; studied and worked at Amsterdam; contributed to foundations of mathematics, logic, topology.

25. M.H.A. Newman, "Hermann Weyl", *Biographical memoirs of Fellows of the Royal Society* 3 (1957), 305 - 328.

26. Albert Einstein (1879 - 1955), German-born American physicist; Nobel laureate (1922); studied at Federal Institute of Technology (Switzerland); worked at Bern, Zürich, Prague, Leyden, Berlin, Institute for Advanced Study (USA); contributed to Brownian motion, photoelectricity, special and general relativity.

27. Arthur Stanley Eddington (1882 - 1944), English astrophysicist;educated at Cambridge; worked at Royal Observatory (Greenwich), Cambridge; contributed to general relativity and stellar astrophysics.

28. Theodor Kaluza (1885 - 1954), German mathematical physicist; studied at Königsberg (now Kaliningrad, USSR); worked at Königsberg, Kiel, Göttingen; contributed to general relativity (the first to introduce a fifth dimension into a unified field theory).

29. The concept of parity in physics is that of mirror symmetry of physical phenomena. For example, a charge moving parallel to an electric current is deflected toward the current by the induced electromagnetic field. In the mirror image of this

phenomenon, the current is reversed and the charge is still deflected toward the current. Parity is conserved in electromagnetic interactions. In 1957, T.D. Lee (1926 -) and C.N. Yang (1922 -) suggested that parity is not conserved for weak interactions. This was confirmed by C.S. Wu (1913 -) and others for beta-decay processes.

30. See, for example, B.L. van der Waerden *Group theory and quantum mechanics*, Springer, Berlin, 1974.

31. Werner Heisenberg (1901 - 1976), German theoretical physicist; Nobel laureate (1932); studied at Munich, Göttingen; worked at Leipzig, Berlin, Göttingen, Munich; contributed to quantum theory.

32. Wolfgang Pauli (1900 - 1958), Austrian theoretical physicist; Nobel laureate (1945); studied at Munich; worked at Göttingen, Copenhagen, Hamburg, Federal Institute of Technology (Switzerland); contributed to quantum theory; predicted existence of neutrino.

33. Paul A. M. Dirac (1902 -), English theoretical physicist; Nobel laureate (1933); studied at Bristol, Cambridge; worked at Cambridge, Institute for Advanced Study (USA); now at Cambridge; contributed to quantum theory; predicted existence of positron.

34. Eugene Paul Wigner (1902 - 1975); Hungarian-born American physicist; Nobel laureate (1963); studied at Institute of Technology (Berlin); worked at Princeton, Wisconsin, Chicago, Clinton Laboratories, Oak Ridge; contributed to quantum theory, nuclear physics.

35. Erwin Schrödinger (1887 - 1961), Austrian physicist; Nobel laureate (1933); studied at Vienna; worked at Vienna, Stuttgart (Germany), Zürich, Berlin, Institute of Advanced Study (Ireland); returned to Vienna after retirement in 1955; contributed to quantum mechanics, theory of colour.

36. Leopold Kronecker (1823 - 1891), German mathematician; studied at Berlin; had independent source of income from own

business; did not hold any university position until 1883 (as professor at Berlin); contributed to number theory, algebra.

37. Richard Dedekind (1831 - 1916), German mathematician; studied at Göttingen (under Gauss); worked at Göttingen, Federal Institute of Technology (Switzerland), Technical Institute (Brunswick); contributed to number theory, analysis.

38. Georg Cantor (1845 - 1918), German mathematician; studied at Zürich, Göttingen, Frankfurt, Berlin; worked at Halle (Germany); contributed to set theory, theory of transfinite numbers.

39. See, for example, Raymond L. Wilder, *Introduction to the foundations of mathematics*, 2nd edition, Wiley, New York, 1965.

40. Hermann Weyl, "Mathematics and logic. A brief survey serving as a preface to a review of 'The philosophy of Bertrand Russell' ", American Mathematical Monthly 53 (1946), 2 - 13. Reprinted in *Gesammelte Abhandlungen*, Band IV.

41. Waring's Problem concerns the determination of the number g(k), which is the smallest value of r for which every positive integer is the sum of r non-negative k^{th} powers. For instance, Lagrange has shown that every positive integer is the sum of 4 squares. The existence of g(k) for each k has been proved by Hilbert in 1909. The value of g(k) is now known for all k except 4 and 5.

42. The Riemann zeta-function is defined by the series $\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots + \frac{1}{n^s} + \ldots$ for complex $s = \sigma + it$, $\sigma > 1$, whose analytic continuation gives a meromorphic function. This has been introduced in the investigation of the number of primes less than a given number.

43. S. M. Ulam, *Adventures of a mathematician*, Charles Scribner's Sons, New York, 1976.

44. Once in four years, the International Mathematical Union holds an International Mathematical Congress at which the prestigious Fields Medal is awarded to one or more mathematicians (below the age of 40) for significant contributions to mathematics.

The recipients of the 1974 Congress at Vancouver were David Mumford of America and Enrico Bombieri of Italy.

45. Quoted on frontispiece of Band I of *Gesammelte Abhandlungen*.

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ANNOUNCEMENT

The International Congress of Mathematicians will be held in Helsinki, Finland, during August 15-23, 1978. Correspondence concerning the Congress should be addressed to

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