## PROBLEMS AND SOLUTIONS

A book-voucher prize will be awarded to the best solution of a starred problem. Only solutions from Junior Members and received before 1 March 1977 will be considered for the prizes. If equally good solutions are received, the prize or prizes will be awarded to the solution or solutions sent with the earliest postmark. In the case of identical postmarks, the winning solution will be decided by ballot.

Problems or solutions should be sent to Dr. Y. K. Leong, Department of Mathematics, University of Singapore, Singapore 10. Whenever possible, please submit a problem together with its solution.

*P9/76. Let P be a probability. Prove that for any two events A and B,

$$|P(A \cap B) - P(A).P(B)| \leq 0.25 .$$

Give an example to show that the upper bound of 0.25 may be attained.

(via Louis H. Y. Chen)

*P10/76. Let $a_0,\ldots,a_4,b_0,\ldots,b_4$ be distinct elements. Let $S \doteq \{ \{a_0,a_1\}, \{a_1,a_2\}, \{a_2,a_3\}, \{a_3,a_4\}, \{a_4,a_0\}, \{a_0,b_0\}, \{a_1,b_1\}, \{a_2,b_2\}, \{a_3,b_3\}, \{a_4,b_4\}, \{b_0,b_2\}, \{b_2,b_4\}, \{b_4,b_1\}, \{b_1,b_3\}, \{b_3,b_0\} \}$. Prove that S is not the disjoint union of subsets $S_1,S_2,S_3$ where each of the $a_i$ and $b_i$ appears at most once in each of the subsets $S_1,S_2$ and $S_3$.

(H. P. Yap)

P11/76. Let $a_0,a_1,\ldots,a_6,b_0,b_1,\ldots,b_6$ be distinct elements. Let $S = \{ \{a_0,a_1\}, \{a_1,a_2\}, a_2,a_3\}, \{a_3,a_4\},$

$\{a_4,a_5\}$, $\{a_5,a_6\}$, $\{a_6,a_0\}$, $\{a_0,b_0\}$, $\{a_1,b_1\}$, $\{a_2,b_2\}$, $\{a_3,b_3\}$, $\{a_4,b_4\}$, $\{a_5,b_5\}$, $\{a_6,b_6\}$, $\{b_0,b_2\}$, $\{b_2,b_4\}$, $\{b_4,b_6\}$, $\{b_6,b_1\}$, $\{b_1,b_3\}$, $\{b_3,b_5\}$, $\{b_5,b_0\}$ }. Find subsets $S_1$, $S_2$, $S_3$ such that $S$ is the disjoint union of $S_1$, $S_2$, $S_3$ and each of the $a_i$ and $b_i$ appears at most once in each of $S_1$, $S_2$, $S_3$.

(H. P. Yap)

*P12/76. For any positive real numbers $x_1, x_2, \ldots, x_n$, Prove that

$$\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i^3\right) \geqslant \left(\sum_{i=1}^{n} x_i^2\right)^2 .$$

(This is given as a mechanics problem).

(via H. N. Ng)

As a supplementary note to P2/76, Dr. H. N. Ng points out that there is a form of Taylor's theorem which probabilists claim to be in calculus books but which turns out to be only in very special books:

Let f be defined on [a,b] and suppose that the derivatives $f'(x), f''(x), \ldots, f^{(n)}(x)$ exist. Let $a \leqslant x_o \leqslant b$; then for all $x \in [a,b]$,

$$f(x) = f(x_o) + \frac{f'(x_o)}{1}(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \ldots + \frac{f^{(n)}(x_o) + \xi(x)}{n!}(x - x_o)^n$$

where

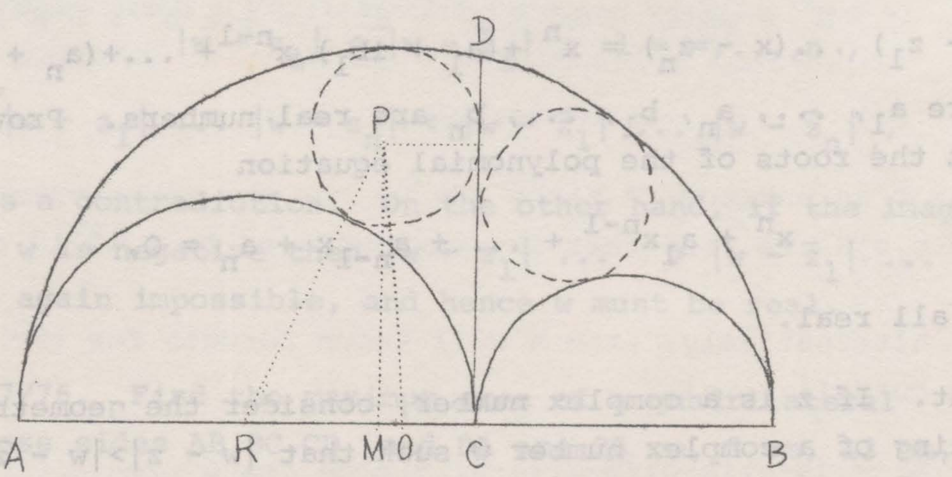
$$\xi(x) \to 0 \text{ as } x \to x_o .$$

Hee Juay Guan has been awarded the prizes for correct solutions to P5, 7/76, and Lim Boon Tiong the prize for a correct solution to P8/76.

Solutions to P5 - P8/76.

*P5/76. (Archimedes' Theorem)

Semicircles are drawn on AB, AC and CB as diameters, where C is any point between A and B. CD is drawn perpendicular to AB. If two circles are drawn such that each touch the larger circle, one of the smaller circles and also CD, prove that these two circles are equal with diameter $CD^2/AB$.




(via Chan Sing Chun)

Let O and R be the mid-points of AB and AC respectively. Let P be the centre of one of the inscribed circles and M the projection of P onto AB. Write AC = 2a, CB = 2b, AB = 2r. Let x be the radius of the inscribed circle with centre P. Then RP = a + x, RM = a - x, OP = r - x, OM = |2a - r - x|. Applying Pythagoras' theorem to $\Delta$sPRM, POM, we have

$$(a + x)^2 - (a - x)^2 = (r - x)^2 - (2a - r - x)^2 .$$

Solving for x, we get

$$x = \frac{a(r-a)}{r} = \frac{ab}{r},$$

i.e. $$2x = \frac{(2a)(2b)}{2r} = \frac{AC.CB}{AB} = \frac{CD^2}{AB}.$$

A similar calculation for the radius of the other inscribed circle gives the same result.

Also solved by Proposer.

P6/76. Let $z_1, z_2, \ldots, z_n$ be n complex numbers whose imaginary parts are positive. Put

$$(x - z_1) \ldots (x - z_n) = x^n + (a_1 + ib_1) x^{n-1} + \ldots + (a_n + ib_n),$$

where $a_1, \ldots, a_n, b_1, \ldots, b_n$ are real numbers. Prove that the roots of the polynomial equation

$$x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n = 0,$$

are all real.

(Hint. If z is a complex number, consider the geometrical meaning of a complex number w such that $|w - z| > |w - \bar{z}|$, where $\bar{z}$ is the conjugate of z.)

(via Ho Soo Thong)

Solution by Proposer.

It is easily seen that if $\bar{z}_1, \ldots, \bar{z}_n$ are the conjugates of $z_1, \ldots, z_n$ respectively, then

$$(x - \bar{z}_1) \ldots (x - \bar{z}_n) = x^n + (a_1 - ib_1)x^{n-1} + \ldots + (a_n - ib_n)$$

where the $a_i$ and $b_i$ are as given in the problem. Hence if $x = w$ is a root of the given polynomial equation, then

$$(w - z_1) \ldots (w - z_n) = i(b_1 x^{n-1} + \ldots + b_n),$$

$$(w - \bar{z}_1) \ldots (w - \bar{z}_n) = -i(b_1 x^{n-1} + \ldots + b_n).$$

It follows that

$$|w - z_1| \ldots |w - z_n| = |w - \bar{z}_1| \ldots |w - \bar{z}_n| .$$

To show that this implies that w must be real, we first make the following observation : if the imaginary part of z is positive, then $|w - z|$ is less than or greater than $|w - \bar{z}|$ according as the imaginary part of w is positive or negative. For if $z = x + iy$, $y > 0$, and $w = u + iv$, then

$$|w - z|^2 - |w - \bar{z}|^2 = -2vy .$$

To complete the proof, suppose that the imaginary part of w is positive. Then by the above remark, we have

$$|w - z_i| < |w - \bar{z}_i| , \quad i = =,\ldots,n ,$$

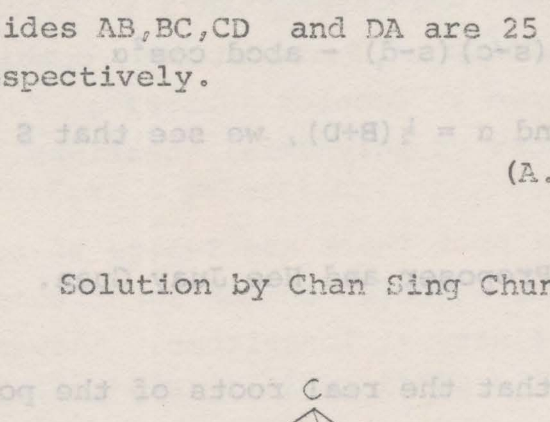
and so $|w - z_1| \ldots |w - z_n| < |w - \bar{z}_1| \ldots |w - \bar{z}_n|$ ,

which is a contradiction. On the other hand, if the imaginary part of w is negative then $|w - z_1| \ldots > |w - \bar{z}_1| \ldots$ . This is again impossible, and hence w must be real.

*P7/76. Find the maximum area of a quadrilateral ABCD whose sides AB, BC, CD and DA are 25 cm, 8 cm, 13 cm, and 26 cm respectively.

(A.D. Villanueva)

Solution by Chan Sing Chun



Let S be the area of the quadrilateral ABCD. Then

$$S = \frac{1}{2} ab \sin B + \frac{1}{2} cd \sin D \ldots \quad (1)$$

Moreover,

$$a^2 + b^2 - 2ab \cos B = c^2 + d^2 - 2cd \cos D,$$

or $$\frac{1}{2} ab \cos B - \frac{1}{2} cd \cos D = \frac{1}{4} (a^2 + b^2 - c^2 d^2) \ldots \quad (2)$$

Squaring (1) and (2), and adding the resulting equations, we have

$$S^2 + \frac{1}{16}(a^2 + b^2 - c^2 - d^2)^2 = \frac{1}{4}(a^2b^2 + c^2d^2)$$

$$-2abcd \cos(B+D).$$

Hence S is maximum if $B+D = 180^\circ$. For the given values of $a,b,c,d$, we find that the maximum area is $2\sqrt{17710}\ \text{cm}^2$.

Alternatively, from a well-known formula for the area of a quadrilateral,

$$S^2 = (s-a)(s-b)(s-c)(s-d) - abcd \cos^2\alpha$$

where $2s = a+b+c+d$ and $\alpha = \frac{1}{2}(B+D)$, we see that S is maximum when $\alpha = 90^\circ$.

Also solved by Proposer and Hee Juay Guan.

*P8/76. Prove that the real roots of the polynomial equation

$$x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0 ,$$

where $a_1, \ldots, a_n$ are integers, are either irrational or integers.

(via C. T. Chong)

Solution by Lim Boon Tiong

A real root of the given polynomial equation is either irrational or rational. Suppose that it is rational, and let it be p/q where p and q are coprime integers. Thus

$$(p/q)^n + a_1(p/q)^{n-1} + \ldots + a_{n-1}(p/q) + a_n = 0 .$$

Multiplying by $q^{n-1}$ gives

$$(p^n/q) + a_1p^{n-1} + a_2p^{n-2}q + \ldots + a_{n-1}pq^{n-2} + a_nq^{n-1} = 0.$$

This implies that $p^n/q$ is an integer and so q = 1 or -1. Hence a rational root of the given equation must be an integer.