NOTES ON MATHEMATICS

6. David Hilbert (1862-1943)

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"The world looked upon David Hilbert as the greatest mathematician living in the first decades of the twentieth century." —— *Hermann Weyl* [1].

There is probably no one who exerted as great an influence on the development of twentieth century mathematics as David Hilbert. His deep insight, his creative power and his broad interest in mathematical sciences made him one of the most impressive contributors to the subject. Yet these attributes and achievements, shared and accomplished at least as much by Poincare [2] and-some would perhaps argue - more by Riemann [3] and Gauss [4], were but part of the reason that gave him lasting radiance and permanent status in mathematics. It was the fortuitous combination of Gottingen the institution and Hilbert the scientific personality that brought together aspring and established mathematicians to the German University where much of the mathematics of this century was subsequently created.

Youth. David Hilbert was born on January 23, 1862, in Wehlau, near Konigsberg [5], the capital of East Prussia. His father was a judge in the city of Konigberg, and David spent his childhood in this city which earned fame in mathematics through the "Konigsberg Bridge Problem" solved a century earlier by Euler [6]. The most **revered** historical figure in the minds and hearts of "Konigsbergers" was the philosopher Immanuel Kant [7], a native son, whose words -

"The greatest wonders are the starry heavens above me and the moral law within me" - must have been repeated to David on many occassions.

David went to school at the age of eight, two years later than the usual starting age. The education included the study of German and Roman scripts, stories from the Bible, and simple arithmetic like subtraction and multiplication.

After two years' preparatory work, he entered the royal Friedrichskolleg Gymnasium [8], reputed to be the best in Konigsberg. The Gymnasium, with its strict emphasis on the study of Latin and Greek, provided young Hilbert with little comfort. Learning by memory was always stressed, and not much opportunity was given for independent thinking. To him this was all the more painful since his linguistic and retentive abilities were perhaps below average. The only subject in which he found delight was mathematics because it required no memorization. It was "easy and effortless". Already at this time he was considered a bit "weird". While "his mother wrote his school essays for him, he could on the other hand explain mathematics problems to his teachers. Nobody really understood him at home." (see [9]). Life took on a brighter colour when he transferred at the beginning of his last gymnasium year (1879) to the Wilhelm Gumnasium which emphasized more mathematics and encourage originality. In the annual report, the teacher noted that David showed in mathematics "a very lively interest and a penetrating understanding" [8] - the first sign of the future mathematician.

In the fall of 1880, Hilbert entered the University of Konigsberg which, by its scientific tradition, was one of the most distinguished universities at the time. The mathematicians who had taught these included Jacobi [10] - the "second best" (after Gauss) mathematician in Europe, Richelot - the successor of Jacobi and the discoverer of Weierstrass

[11], and Franz Neumann [12] - the founder of the first institute of theoretical physics at a German university and the originator of the seminar format. But the greatest benefit which Hilbert derived from the University was the chance meeting with two young mathematicians Minkowski [13] and Hurwitz [14] who were destined to become his life long friends and to influence most greatly his mathematical career.

In the spring of 1882, the shy and boyish, seventeen year-old Hermann Minkowski came to Konigsberg after studying for a year at the University of Berlin. Despite his youthful age, he had already won a mathematical prize at Berlin and had just concluded a deep mathematical investigation into a problem in the theory of numbers. In the year that he went to Konigsberg, the Paris Academy was considering this work for the Grand Priz des Sciences Mathematiques. The announcement that he was a co-winner (with the British mathematician Henry Smith) of the Grand Prix came in the spring of 1883. Minkowski, now 18, caused an instant sensation in the city of Konigsberg. Hilbert, whose mathematical abilities were yet to show, became friends with Minkowski despite his father's caution on impertinence by associating with "such a famous man." The two found common traits in each other, discussed and pursued mathematics together, and soon shared the conviction that "every definite mathematical problem must necessarily be susceptible of an exact settlement, either in the form of an actual answer to the question asked, or by the proof of the impossibility of its solution and therefore the necessary failure of all attempts". 9

In the spring of 1884, 25-years-old Adolf Hurwitz arrived at Konigsberg from Gottingen. He was to take up the job of an Extraordinarius [15]. Immediately Hilbert "found the new teacher 'unpretentions in his outward appearance' but saw that 'his wise and gay eyes testified as to his spirit.'" [9]. The two students soon set up a close relationship with the young teacher and it was around this time that the legendary 'apple tree walk', which they had every afternoon at five o'clock, began. This was how mathematical knowledge was instilled into the minds of the two aspring youths. In his obituary on Hurwitz, Hilbert recalled that "on innumerable walks, at times undertaken day by day, we browsed in the course of eight years through every corner of mathematical science. We engrossed ourselves in the actual problems of the mathematics of the time; exchanged our newly acquired understandings, our thoughts and scientific plans; and formed a friendship for life." And "we did not believe that we would ever bring ourselves so far."

In late winter, 1885, Hilbert received his Doctor of Philosophy degree, having written a dissertation in invariant theory and passed an oral examination. That summer, Minkowski also received his doctor's degree but left immediately for his year in the army. As Hilbert was not called up for military service, Hurwitz suggested that he went to the University of Leipzig to work under the renowned figure Felix Klein [16]. Klein, who was then only 36, had accomplished distinguished mathematical works. He became an Ordinariut (a full professor (see [15])) at the young age of 23. His achievements in the theory of automorphic functions - combining geometry, number theory, group theory, invariant theory, and algebra under one roof - had been unrivaled until a French mathematician Henri Poincare unleashed his series of papers on that subject. This started an unofficial competition between the two and led eventually to a deep mental depression and nervous breakdown of Klein.

Klein had just recovered from this misfortune when Hilbert visited Leipzig. The admiration that they had for each other was apparent. "When I heard him lecture," said Klein years later, "I know immediately that he was the coming man in mathematics." [9]

In the spring of 1886, Hilbert went to Paris under the suggestion of Klein, to "master all the results of (the young talents of the French) in order to go beyond them." [9] That same year, Klein was offered and accepted a professorship at the University of Gottingen - the institution which enshrined, above all else, the brilliant mathematics of Gauss, Dirichlet [17] , and Riemann.

The next year Hilbert returned to Konigsberg to prepare himself for "Habilitation." (see [15]) He was awarded this title within a month, and began to lecture on various topics in mathematics to a small group of students who cared to pay him by attending his classes. He was now working on the famous problem in invariant theory called 'Gordan's Problem' - to look for a finite basis in an infinite system of forms.

Fame. In December, 1888, Hilbert published a complete solution of Gordan's Problem. Yet the method by which he obtained the proof was so unconvetional and revolutionary that it was branded variously as being uncomfortable, sinister, and weird. Gradual acceptance of the proof did come, however. The founder of invariant theory, Arthur Cayley [18] , wrote to him, "I think you have found the solution of a great problem." Gordan [19] , the originator of the problem, conceded that "Herr Hilbert's proof was completely correct," after an initial outcry that Hilbert's proof was theology and not mathematics. "I have convinced myself that theology also has its merits." [9]

During the next two years Hilbert continued to work on invariant theory. By 1892, his contributions to the theory virtually closed the final chapter on the subject. "I believe the most important goals of the theory of function fields generated by invariants have been obtained." [9] He wrote to Minkowski, who was now at the University of Bonn, that "I shall definitely quit the field of invariants."

The young mathematician who has recently established a position in the German mathematical community now looked to mathematics beyond the theory of invariants. Next target : algebraic number theory.

The next three years saw some important changes in Hilbert's life. He married and fathered a child. Hurwitz was offered a full professorship at the Swiss Federal Institute of Technology in Zurich and Hilbert was promoted to the rank of Extraordinarius which Hurwitz had left vacant. Minkowski also became an Extraordinarius at Bonn. Suddenly the two was separated and Konigsberg, to Hilbert at least, transformed into a pure mathematics Antarctic. No longer would there be any daily "apple tree walk."

Around this time Hilbert began his mathematical work on algebraic number theory. While Gauss considered classical number theory to be the queen of mathematics, Hilbert, having the fortune of seeing how that theory developed into algebraic number theory under the efforts of Dirichlet, Kummer [20], Dedekind [21], and Kronecker [22], considered number theory to be a structure of rare beauty and harmony. Already before Hurwitz left for Zurich, the discussions of their daily "apple tree walk" had been concentrated on algebraic number theory. And now he embarked on this new field by presenting a new proof of the unique factorization of rings of integers into prime ideals in number fields.

In 1893, the Deutsche Mathematiker - Vereinigung (German Mathematical Society) commissioned Hilbert and Minkowski to write a report on number theory to be completed in two years. Hilbert welcomed this as an opportunity to lay a foundation for the theory which hitherto had been marked with an overabundance of diverse notations and theorems whose proofs contained gaps. But more than that he saw that a foundation for the theory was a prerequisite for a deeper investigation into algebraic numbers. For this purpose he read virtually every paper on number theory published since the time of Gauss, looking for proofs of theorems "whose principles are capable of generalization and the most useful for further research." [23]

It was at this time that the 31-year-old mathematician was promoted to the rank of Ordinariut. But the rising star in mathematics could not have been satisfied with a professorship in Konigsberg which, despite its outstanding tradition, remained outside the scientific mainstream in Germany. In the winter of 1894, the call from Gottingen finally arrived : Weber [24] was leaving for Strassburg and Klein had proposed that Hilbert be the successor to the professorship. "You are the man whom I need as my scientific complement because of the direction of your work and the power of your mathematical thinking and the fact that you are still in the middle of your productive years. I am counting on it that you will give a new inner strength to the mathematical school here, which has grown continuously and, as it seems, will grow even more."

This had been a dream come true for Hilbert. Immediately, he replied that the letter "has surprised me in the happiest way. It has opened up a possibility for the realization of which I might have hoped at best in the distant future and as the final goal of all my efforts ... ." [9]

<u>Gottingen</u>. The University of Gottingen, beautifully situated in a small town, now welcomed Hilbert in the spring of 1895 - almost 100 years after Gauss in anticipation of the great scientific contributions which he was to offer. To the students his unpretentions appearance, unlike Klein's dignified aloofness to which they had been accustomed, made him seem to be not at all like a professor. Students began to be impressed by his lectures which, though tripped up by details and gaps quite frequently, saw many new ideas developed rather extemporaneously and unexpectedly.

The work on the number theory report continued. But Minkowski's progress on his portion was slow, and by 1896 when Hilbert had finished his, they agreed that perhaps the finished portion should be published first. The monumental report *Die Theorie der algebraischen Zahlkorper* appeared in 1897. Hermann Weyl wrote an appraisal of the work : "What Hilbert accomplished is infinitely more than the vereinigung could have expected. Indeed his report is a jewel of mathematical literature. Even today, after almost fifty years, a study of this book is indispensable for anybody who wishes to master the theory of algebraic numbers." [25]

During the next two years, Hilbert talked about nothing but number fields. His last and most important paper (published in 1899) in this area, dealing with the theory of relative abelian extensions of number fields, formulated the basic facts about "class fields." This paper, in contrast to his work on invariant theory, opened up the field of algebraic numbers. Some of the world's leading mathematicians in later decades worked on the problems posed in the paper. One need only mention names like Takagi [26] ("existence theorem" for abelian extensions corresponding to generalized ideal class groups), Hasse [27] (the "simple algebra" theoretic approach to class field theory), Artin [28] (the "Artin Reciprocity Law"), Chevalley [29] (the concept of ideles and the class field theory for infinite abelian extensions), and Hecke 30 (the function-theoretic approach to number theory).

During the period 1898 - 1902, Hilbert's subject of study was the foundations of geometry. He was seized by the idea of axiomatics. Under the axiomatic approach, geometry became a hypothetical deductive system. It was not necessary to know what a point, a line, or a plane was. All that was needed was to set up axioms for geometry which satisfy the conditions of consistency (so that no contradictory theorems be proved), independence (so that no axiom be redundant), and completeness (so that all true theorems could be proved). One could then argue on a purely formal basis and derive thereby theorems which would apply equally well to 'tables, chairs, and beer mugs.' Using this, Hilbert showed that non-Euclidean geometry was as consistent as Euclidean geometry and as consistent as the arithmetic of real numbers.Poincare commented on Hilbert's *Foundations of geometry*, a book that became an instant classic, that "he has made the philosophy of mathematics take a long step forward ... " Indeed Hilbert's contributions to the axiomatic approach were not confined to geometry. He had set an influence whcih engulfed his and later generations' attitude towards the whole of mathematics.

Today one finds the axiomatic spirit permeating through every branch of mathematics : from mathematical logic-axiomatics being the object of study, to algebra - axioms for groups, rings, fields, ... , to topology - the Steenrod-Eilenberg axioms for homology, to analysis - axioms for Hilbert space, Banach space, ... Even his investigations in the field of physics in later life were conceived in the axiomatic spirit.

While the work on geometry went on, Hilbert presented in 1899 a result which virtually salvaged the famous Dirichlet Principle (concerning solutions of what is known as the boundary value problem of the Laplace equation), freely used by Riemann in his work but later put into disgrace by Weierstrass who pointed out that the principle was not always valid. That winter, he lectured on the calculus of variantions, a sign that the 37 year old professor's mathematical interests had now diversified further than his early days at Konigsberg. His lectures continued to bring joy and invite admiration. With Minkowski and Hurwitz absent, Hilbert now carried on the tradition of "apple tree walk" with his students : a weekly "seminar walk" into the woods near Gottingen. Lively mathematical discussions dominated throughout. Max von Laue [31] , a future Nobel laureate who was then a student, once said that "this man lives in my memory as perhaps the greatest genuis I ever laid eyes on." [9]

The future of mathematics. One of the greatest honors a mathematician can receive in recognition of his achievements is an invitation to deliver a major address at the International Congress of Mathematicians. In the year 1900, such an honor was extended to Hilbert whose fame in mathematics was then equaled only by that of Poincare. The lecture that he gave went down in history as one that anticipated, predicted, and indeed shaped the future of mathematics. Today, more than seventy years after the delivery of the lecture, we see in retrospect that a large part of contemporary mathematics was derived from the lecture.

"History teaches the continuity of the development of science. We know that every age has its own problems, which the following age either solves or casts aside as profitless and replaces by new ones. If we would obtain an idea of the probable development of mathematical knowledge in the immediate future, we must let the unsettled question pass before our minds and look over the problems which the science of today sets and whose solution we expect from the future.

"The deep significance of certain problems for the advance of mathematical science in general and the important role which they play in the work of the individual investigator are not to be denied. As long as a branch of science offers an abundance of problems, so long is it alive; a lack of problems foreshadows extinction or the cessation of independent development.

"The conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpentual call : There is a problem. Seek its solution. You can find it by pure reason, for in mathematics there is no *ignorabimus*." [32]

Twenty-three unsolved problems were posed [32] . These problems have played an eminent role in the development of mathematics ever since, and "we mathematicians have often measured our progress by checking which of Hilbert's questions have been settled 33 ." The list of names of contributors to their solutions reads like a roll call of leading mathematicians, past and present.

Each of the problems was marked by the simplicity of its statement and the significance of its consequences. The continuum hypothesis [34] (Problem 1), the consistency of arithmetic [35] (Problem 2), mathematical treatment of the axioms of physics (Problem 6), the transcendence of certain numbers [36] (Problem 7), the Riemann hypothesis [37] (Problem 8), reciprocity law (Problem 12) [38] , minimal surfaces 39 (Problem 20) : all these, and others, have proved to be privotal in the development of modern mathematics. Indeed Hilbert would be long remembered for just presenting these problems at the Congress. And who can deny that only men of his deep insight and calibre would be able to see these problems as crucial for the evolution of mathematics? [40]

At the beginning of this century, Hilbert lectured on integral equations and potential theory. He was now so famous that students from as far away as North America came to attend his lectures. The recently founded Bulletin of the American Mathematical Society featured regularly mathematical courses offered at the University of Gottingen. Foreign academies elected him to membership, and he was awarded the title of Geheimrat, something like the British Knighthood. Included among those who went to Gottingen in answer to the sweet notes of the flute of the Pied Piper were Erhard Schmidt [41] , Caratheodory [42] , Takagi, Blumenthal [43] , Zermelo [44] , Max Born [45] , and Hermann Weyl who later wrote that as soon as he arrived at Gottingen, "the doors of a new world swung open for me, and I had not sat long at Hilbert's feet before the resolution formed itself in my young heart that I must by all means read and study whatever this man had written." [9] Gottingen was considered "the shrine of pure thought," and mathematics students were advised to pack their luggage and go to Gottingen.

Short Reunion. In 1902 Minkowski was offered a professorship at Gottingen through a skillful manoeuvre conducted by Hilbert. After a lapse of some years, the two friends resumed their weekly walk. Now in the company of Minkowski and at Minkowski's suggestion, Hilbert took up the study of physics - a subject which, second only to number theory, had been occupying Minkowski's mind since the time of teaching at Zurich. Their reunion, however, turned out to be short-lived. In January 1909, amidst a display of intellectual fireworks by these two distinguished professors in their discussions of science, mathematics, art, politics and philosophy, Minkowski was striken with a violent attack of appendicitis. A few days later he died, as Hilbert had written to Hurwitz, "in the full possession of his vital energy, in the middle of his most joyful work, at the height of his scientific creativity." The following day, Hilbert announced the death of Minkowski at a class and wept.

To Gottingen, Minkowski's departure was a great loss. To Hilbert, the loss of a dear friend was so great that for some time he was unable to control his grief.

"He was for me a gift of the gods - such a one as would seldom fall to a person's lot - and I must be thankful that I so long possessed it." [8]

Foundations of Mathematics . Hilbert continued to work on physics through the first world war. Around this time Emmy Noether [46] arrived at Gottingen, and became the first female Privatdozent in the history of a German university, despite initial opposition from the university administration. Noether represented a new genration of Gottingen mathematicians, and she was destined to make an indelible mark on the mathematics of the twentieth century. Among the students who came to her were van der Waerden [47] from Holland and Emil Artin from Austria. The three of them literally shaped what we know today as "modern algebra."

But Hilbert's interest had now shifted to the foundations of mathematics. As early as 1900, he had showed that geometry was as consistent as the arithmetic of numbers. What about the consistency of the latter? He raised this as the second on the list of his 23 famous problems. And what about set theory, the subject which was developed by Georg Cantor [48] in the late nineteenth century and which had met many objections and criticisms, notably from Kronecker and Poincaré, because of the apparently unresolvable antinomies present within it [49] ?

Hilbert's philosophy of mathematics had always been one of optimism. He had long opposed Kronecker's idea that the integers form the foundation of arithmetic and that construction by means of a finite number of integers was therefore the only possible criterion of mathematical existence. Similarly, he had always believed that the consistency of number theory, hence the mathematics, could be given a wholly rigorous mathematical proof. At the 1904 International Congress of Mathematicians held in Heidelberg, he propounded these ideas and insisted that mathematics "can and must" have a foundation. Thus unlike Kronecker, who proposed to put great restrictions on mathematics and mathematical methods, Hilbert believed, with Cantor, that the essence of mathematics is its freedom, without which the continuing evolutionary process of the subject would be jeopardized.

It is worthwhile to note that the ideas put forth by Hilbert in 1904 was a particularly bold one. The general sentiment towards set theory had always been less than sympathetic - partly because of the strangeness of Cantor's ideas, partly because of the attacks launched by Kronecker and Poincaré, and partly because of the various antimomies discovered. In 1895 Minkowski had lectured on Cantor's theory of the infinite, and that again had been a bold move. "Later histories will call Cantor one of the deepest mathematicians of this time," Minkowski said.[9]

By the end of the second decade of the 20th Century, the state of "foundational studies" was in disarray. Russell [50] and Whitehead [51] had published their *Principia Mathematica* in the attempt to found mathematics on logic, an attempt which was not successful and received mixed reactions. Brouwer [52] had abandoned his distinguished work on topology and expounded in all fervour his philosophy of mathematics in the form of *intuitionism*, denouncing even one of the most basic principles of (two-valued) logic – the law of excluded middle [53]. To Hilbert, if Brouwer's views were strictly adhered to, a major part of classical mathematics would have to be rewirtten or abandoned and Cantor's theory of infinite sets would have to be given up – and such "mutilation" of mathematics was unacceptable to him. To solve the foundational problem in a way that was satisfying, he proposed to formalize mathematics by introducing symbols for concepts and relations, and axioms for general mathematical theories and deducing, via principles of logic, theorems involving the given symbols. This is certainly reminiscent of his treatment of the foundations of geometry. But in this case his programme was much more ambitious : from such formalism he proposed to prove firstly the consistency of arithmetic, of set theory, eventually of mathematics, and secondly their completeness. One could see from here the drastic difference of attitude on the foundations of mathematics held by the two leading mathematicians of the time : Poincare had sided with Kronecker and referred to set theory as "an interesting pathological case." He also predicted that "later generations will regard Cantor's theory as a disease from which one has recovered [54]." Hilbert on the other hand praised that Cantor's work was "the most astonishing product of mathematical thought, one of the most beautiful realizations of human activity in the domain of the purely intelligible," and "no one shall expel us from the paradise which Cantor created for us." [55]

During the year 1920-30, Hilbert and his students — Ackermann [56], and Bernays [57], together with the young von Neumann [58], evolved what is known as *pröof theory* to study methods of establishing the consistency of formal systems. Their work demonstrated the consistency of a number of formal systems, but then in 1931, Godel [59] dealt a death blow to the Hilbert program by showing that any axiomatization of number theory was necessarily incomplete, and that the consistency of number theory could not be proved within the axiomatization if it were consistent. A startling corollary of Godel's work was that the intuitively certain (for example, the consistency of number theory) goes further than mathematical proof.

Despite such setbacks, a member of the Hilbert school, Gentzen [60] did show in 1936, by means of transfinite induction, the consistency of number theory and therefore of set theory, but this went beyond what was allowed in Hilbert's program. Under the efforts of Zermelo, Frenkel [61], von Neumann, Godel, and recently Cohen [62], set theory flourished and became an accepted discipline of mathematics, thus giving Hilbert one triumph over Poincare.

<u>Eng of an Era</u>. Hilbert's health started to deteriorate in the mid 1920's. By 1928, he was 66 and Hilbert's career was almost over. In that year, through the efforts of Courant [63], a new building for the Mathematical Institute of the University of Gottingen was completed. Two years later, he retired from the professorship which he held for the past 35 years. His "farewell lecture" was, once agakn after more than 40 years, on the theory of invariants, the subject which brought him initial fame. A street was named "Hilbert Strasse" in honor of him. And the city of Konigsberg awarded him an honorary citizenship.

A collection of Hilbert's work was edited and the first volume, consisting entirely of number theory, was presented to him on his seventieth birthday. A party in the Mathematical Institute was held that evening, and a procession of students carrying torches marched to the building and shouted for Hilbert. That was the highest honor that a professor could receive from students.

The year of his seventieth birthday marked, unfortunately, the beginning of the dissolution of the Hilbert School at Gottingen. In the following January, Adolf Hilter was appointed chancellor of Germany. Immediately came the order that all Jews holding any sort of teaching positions should be removed from the universities. Within a few months, mathematicians and physicists at Gottingen packed up and left. By late summer, almost everyone was gone. Courant, Landau [64] , Noether, Bernays, Born, Weyl and many others either emigrated to the United States or Switzerland, or remained in Germany but without jobs.

Hilbert started to lead a secluded life. His memories began to lapse. The visitors who came to talk to him about mathematics found him responding attentively, but more politely than intelligently. Mathematical activities at the University were now minimal. When once asked by a Nazi minister about mathematics at Gottingen, he replied, "there is really none anymore." [9]

In the spring of 1943, Hilbert died at the age of 81.

"The report of Hilbert's death brought up again the whole Gottingen past for me," Hermann Weyl wrote from Princeton, U.S.A., to the widow of Minkowski, "I had the great luck to grow up in the most beautiful years when Hilbert and your husband both stood in the prime of their power. I believe it has very seldom happened in mathematics that two men exercised such a strong and magic influence on a whole generation of students. It was a beautiful, brief time." [9]

Thus the brilliant radiance that emitted from the

Univeristy of Gottingen shone for almost half a century. It slowly flickered and eventually disappeared from the dark sky over Nazi Germany. What Hilbert bequeathed to the mathematical world, however, will remain for many centuries to come.

Hilbert knew no prejudice - be it age, racial, national, or sexual. To him there were only two kinds of scientists : those who produced works of recognized worth, and those who did not. He was always ready to accept new ideas and was most impressed by what he did not yet know. In him there was no snobbish attitude of pretended indifference, often manifested by mathematicians of much lesser quality ("condemning everything that they do not know," as Hilbert once said). He worked enormously hard, and often quoted the famous saying "genius is industry." He was furthermore a spiritual leader. As Hermann Weyl said, "Under the influence of his dominating power of suggestion one readily considered important whatever he did; his vision and experience inspired confidence in the fruitfulness of the hints he dropped." 65 His openness and eagerness to learn and work attracted scientists of the highest calibre. With him at the center, one found around him a circle of dedicated scholars who in joint competitive aspiration of related aims stimulated one another. The Hilbert influence was universal. The tremendous development of American mathematics since the second world war owed not only to men who went to the New World after Hitler's persecution of Jews, but also to the precedent set by Hilbert's style of scientific endeavours at Gottingen, where free-spirited discussions dominated throughout and research, not teaching, was the prime objective. The establishment of the Institute for Advanced Study in Princeton in 1931 was a testimony to Hilbert's impact on Ametican mathematics.

More than anything else, Hilbert was an optimist. He believe that each and every mathematical statement could either be established or its impossibility demonstrated. This belief he repeated many times in the words "Wir mussen wissen. Wir werden wissen." ("We must know. We shall know.")

Notes

[1] Hermann Weyl (1885-1955), German-American mathematician; studied and taught at Göttingen; emigrated to America to become one of the founding members of the Institute for Advanced Study in Princeton; contributed to function theory on Riemann surfaces, unified field theory of gravitation and electricity, and eigenvalue problems of differential equations.

[2] Henri Poincaré (1854-1912), leading French mathematician of 19th century; studied at Ecole Polytechnique and taught at the University of Paris; see C. T. Chong, "Notes on Mathematicians : 4. Henri Poincaré," this *Medley*, vol. 4, No. 1, 1976, 13-34.

[3] Bernhard Riemann (1826-1866), German mathematician; studied at Göttingen and became the successor of Dirichlet there; contributed to function theory with far-reaching consequences in number theory, topology and differential geometry.

[4] Carl Friedrich Gauss (1777-1855), German mathematician; studied and taught at Göttingen; contributed significantly to almost every branch of pure and applied mathematics; See C. T. Chong, "Notes on Mathematicians : 1. Carl Friedrich Gauss", this *Medley*, vol 3, No. 1, 1975, 6-10.

[5] Now Kaliningrad; became a Russian city after the Potsdam Convention in 1945.

[6] Leonhard Euler (1707-1783), Swiss-Russian mathematician; probably the most prolific mathematician in history; contributed to number theory, analysis, astronomy, and solved the "Königsberg Bridge Problem" which marked the birth of topology.

[7] Immanuel Kant (1724-1804), a leading philosopher of the 18th century; propounded the idea of "synthetic - analytic" distinction in the theory of knowledge.

[8] German term for a school which offers a "mental gymnastic" for school boys. The study of Latin and Greek was emphasized. Students, upon graduation, would enter university.

[9] Constance Reid, *Hilbert*, Springer-Verlag, New York, 1970.

[10] Carl Jacobi (1804-1851), German mathematician; contributed to elliptic and abelian functions, number theory, dynamics, algebra.

[11] Karl Weierstrass (1815-1897), German mathematician; contributed to the theory of functions; taught at a high school and talent was "discovered"; later taught at the University of Berlin.

[12] Franz Neumann (1798-1895), German mathematician; taught at Königsberg; also contributed to physics and minerology; divised the first mathematical theory of electrical induction.

[13] Hermann Minkowski (1864-1909), German mathematician; taught at Bonn, Zürich, and Göttingen; contributed to number theory and mathematical physics.

[14] Adolf Hurwitz (1859-1919), German mathematician; taught at Königsberg, Zürich; contributed to number theory and analysis.

[15] In the pre-war German academic system, a doctor of

philosophy is required to produce another original piece of work for the title of "Habilitation." He would then be awarded the rank of "Privatdozent" with the privilege to lecture in the university without pay. He would, however, collect fees from students attending his lectures. Upon recognition of his work and abilities, he would receive the salaried post of an Extraordinarius (assistant professor). The final goal would be an Ordinariant (full professorship).

[16] Felix Klein (1849-1925), German mathematician; taught at Leipzig and Göttingen; contributed to the theory of automorphic functions and Riemann surfaces.

[17] P. G. Lejeune Dirichlet (1805-1859), German mathematician; studied and succeeded Gauss at Göttingen; contributed to analysis and number theory.

[18] Arthur Cayley (1821-1895), English mathematician; taught at Cambridge; contributed to the theory of algebraic invariants.

[19] Paul Gordan (1837-1921), German mathematician; contributed to the theory of invariants.

[20] Ernst Eduard Kummer (1810-1893), German mathematician; taught at Berlin; contributed to number theory by inventing "ideal numbers" and giving partial solutions to Fermat's Last Theorem; originator of modern algebraic number theory.

[21] Richard Dedekind (1831-1916), last student of Gauss at Göttingen; taught at Göttingen, Zürich, and later at a technical high school for fifty years; contributed to number theory and the foundations of analysis.

[22] Leopold Kronecker (1823-1891), German mathematician; taught at Berlin; contributed to number theory; once

remarked that he had spent more time thinking about philosophy than about mathematics; a controversial figure who was opposed to the mathematics of Weierstrass in analysis, and to that of Dedekind and Cantor in the foundations of analysis and set theory.

[23] Cf. preface to his *Theorie der algebraischen Zahlkörper*.

[24] Heinrich Weber (1842-1913), German mathematician; taught Hilbert at Königsberg and later taught at Göttingen; contributed to the theory of numbers.

[25] Hermann Weyl, "David Hilbert and his mathematical work", *Bulletin of the American Mathematical Society*, vol. 5, 1944, 612-654.

[26] Teiji Takagi (1875-1960), Japanese mathematician; studied and taught at Göttingen; also taught at Tokyo; contributed to algebraic number theory.

[27] Helmut Hasse (born 1898), German mathematician; studied and taught at Göttingen; contributed to algebraic number theory.

[28] Emil Artin (1898-1962), German mathematician; studied at Göttingen; taught at Hamburg and later at Princeton, U.S.A.; returned to Hamburg during his last years; contributed to algebra, topology, and number theory.

[29] Claude Chevalley (born 1909); French mathematician; studied under Artin at Hamburg; a founder member of Bourbaki (a group of distinguished French mathematicians who published mathematical treatises under this pseudonym); taught at Princeton, Columbia, and now at University of Paris, contributed to algebraic number theory, algebraic geometry.

[30] Erich Hecke (1887-1947), German mathematician; assistant to Hilbert at Göttingen and later taught

at Göttingen, Hamburg; contributed to number theory.

[31] Max von Laue (1879-1960), German physicist; studied at Göttingen and taught at Zürich, Berlin; received Nobel Prize in physics (1914) for work on diffusion of X-rays in crystals.

[32] David Hilbert, "Mathematical problems", *Bulletin of the American Mathematical Society*, vol 8, 1902, 437-445, 478-479. Reprinted in *Mathematical developments arising from Hilbert problems* (edited by Felix Browder), Proceedings of Symposia in Pure Mathematics, vol. 28, American Mathematical Society, 1976.

[33] Hermann Weyl, "A half-century of mathematics," *American Mathematical Monthly*, vol. 58, No. 8, 1951, p.125.

[34] The continuum hypothesis, introduced by Cantor, states that every uncountable set of real numbers can be put into one-to-one correspondence with the set of all real numbers.

[35] The problem of the consistency of arithmetic seeks to establish by mathematical argument the consistency of the theory of numbers.

[36] For example, is the number $2^{\sqrt{2}}$ transcendental (or at least irrational)? In other words, can it satisfy an equation of the type $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_o = 0$ where $a_n, a_{n-1}, \ldots, a_1, a_o$ are integers?

[37] Riemann had, in the course of his work on the distribution of prime numbers, introduced the zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots + \frac{1}{n^s} + \ldots$$

where s is a complex number. He had conjectured that the only time when $\zeta(s) = 0$ with s lying on the right hand side of the complex plane is when the real part of s is equal to ½. This conjecture, now known as the Riemann Hypothesis, was once called by Hilbert "the most important problem in mathematics." Since Hilbert's time, tremendous progress had been made on this problem. André Weil (born 1906), another member of Bourbaki; now at the Institute for Advanced Study, Princeton showed (1948) that the Riemann Hypothesis is true in the case of function fields over a finite field; Norman Levinson (1912-1975) of the Massachusetts Institute of Technology showed (1973) that at least one third of the zero's of $\zeta(s)$ have real parts equal to ½. Somewhat earlier the young Belgian mathematician Pierre Deligne (born 1946), now at the Institute des Hautes Études Scientifiques in Paris, proved the famous Weil conjecture which is essentially the Riemann Hypothesis for varieties over finite fields. Cf H. M. Edwards, *Riemann's zeta function*, Academic Press, 1974, for a history of this hypothesis.

[38] The reciprocity law, originated from Gauss, states certain relationships between integers and integral solutions of quadratic equations. Since Hilbert's time, more and more "general" reciprocity laws have been found, evolving into what is now called "class field theory" which is among the deepest subjects in mathematics.

[39] The minimal surface problem seeks to find a surface of least area among all surfaces having a prescribed boundary.

[40] For more details of recent developments of the Hilbert problems, see the symposium volume cited in [32] .

[41] Erhard Schmidt (1876-1959), German mathematician; studied at Göttingen and taught at Berlin; contributed to analysis, topology, number theory.

[42] Constantin Carathéodory (1873-1950), Greek-German mathematician; studied at Göttingen and taught at Berlin, Munich; contributed to function theory.

[43] Otto Blumenthal (1876-1944), German mathematician; was Hilbert's oldest student at Göttingen; contributed to number theory.

[44] Ernest Zermelo (1871-1953), German mathematician; studied at Göttingen and was one of the originators of the Zermelo-Fraenkel axiomatic set theory.

[45] Max Born (1882-1970), German physicist; studied and taught at Göttingen; co-winner of Nobel Prize for physics (1954) for his statistical studies of wave functions.

[46] Emmy Noether (1882-1935), German mathematician; studied at Göttingen; taught at Göttingen and Bryn Mawr, U.S.A.; a major contributor to "modern algebra."

[47] B. L. Van der Waerden (born 1903), Dutch-German mathematician; studied and taught at Göttingen; contributed to algebra.

[48] Georg Cantor (1845-1918), German mathematician; best known for his invention of set theory; suffered several nervous breakdowns due to opposition of several mathematicians to his theory; died in a mental hospital.

[49] Antinomies, for example, of the form "the set of all sets which are not members of themselves."

[50] Bertrand Russell (1872-1970); with Whitehead considered to be two of the most influential British philosophers of the present cnetury; studied and taught at Cambridge, Harvard; received Nobel Prize for literature in 1952.

[51] Alfred North Whitehead (1861-1947), taught Russell at Cambridge; later became professor at Harvard.

[52] L. E. J. Brouwer (1881-1966), Dutch mathematician; taught at the University of Amsterdam; contributed to topology and the philosophy of mathematics.

[53] The law of excluded middle states that, independently of human knowledge, every statement is either true or false.

[54] Henri Poincaré, *Oeuvres (Collected Works)*, vol. 5, 19-23.

[55] David Hilbert, "Über das Unendliche" ("On the Infinite"), *Mathematische Annalen*, vol. 95, 1926.

[56] Wilhelm Ackermann (1896-1962), German mathematician, studied at Göttingen; contributed to mathematical logic.

[57] Paul Bernays (born 1888), German mathematician; studied and taught at Göttingen; contributed to mathematical logic.

[58] John von Neumann (1903-1957), Hungarian-American mathematician; contributed to various branches of pure and applied mathematics and computer science; one of the founding members of the Institute for Advanced Study, Princeton; see Y. K. Leong, "Notes of Mathematicians : 3. John von Neumann", this *Medley*, vol. 3, No. 3, 1975, 90-106.

[59] Kurt Gödel (born 1906), Austrian-American mathematician; studied at Vienna and now a permanent member of the Institute for Advanced Study, Princeton; showed the consistency of the continuum hypothesis; work on logic created great impact on the foundations of mathematics.

[60] Gerhard Gentzen (1909-1945), German mathematician; studied at Göttingen; contributed to mathematical logic.

[61] Abraham A. Fraenkel (1891-1966), Israeli mathematician;

taught at the Hebrew University of Jerusalem; co-inventor of the Zermelo-Fraenkel set theory.

[62] Paul J. Cohen (born 1934), American mathematician; studied at Chicago and now professor at Stanford; awarded Fields Medal (1966) by the International Congress of Mathematicians for the proof of the independence of the continuum hypothesis; also contributed to harmonic analysis and partial differential equations.

[63] Richard Courant (1888-1972), German-American mathematician; studied and taught at Göttingen, New York; emigrated to United States during the war; contributed to applied mathematics and minimal surfaces.

[64] Edmund Laudau (1877-1938), German mathematician; taught at Göttingen; contributed to number theory.

[65] Hermann Weyl, "Obituary of David Hilbert," in Royal Society of London, *Obituary Notices of Fellows*, vol. 4, No. 13, 1944, 547-552.