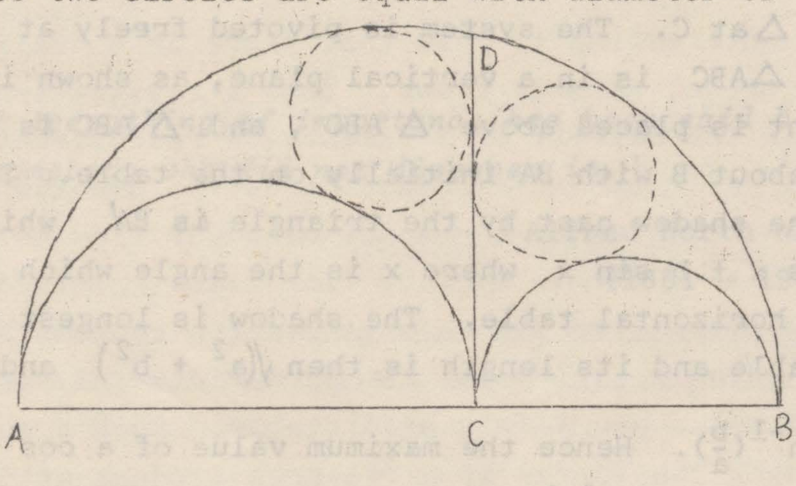
PROBLEMS AND SOLUTIONS

A book-voucher prize will be awarded to the best solution of a starred problem. Only solutions from Junior Members and received before 1 November 1976 will be considered for the prizes. If equally good solutions are received, the prize or prizes will be awarded to the solution or solutions sent with the earliest postmark. In the case of identical postmarks, the winning solution will be decided by ballot.

Problems or solutions should be sent to Dr. Y.K.Leong, Department of Mathematics, University of Singapore, Singapore 10. Whenever possible, please submit a problem together with its solution.

*P5/76. (Archimedes' Theorem)

Semicircles are drawn on AB, AC and CB as diameters, where C is any point between A and B. CD is drawn perpendicular to AB. If two circles are drawn such that each touch the larger circle, one of the smaller circles and also CD, prove that these two circles are equal with diameter CD^2/AB.



(via Chan Sing Chun)

P6/76. Let $z_1, z_2, \ldots, z_n$ be n complex numbers whose imaginary parts are positive. Put

$$(x - z_1) \ldots (x - z_n) = x^n + (a_1 + ib_1)\, x^{n-1} + \ldots + (a_n + ib_n),$$

where $a_1, \ldots, a_n, b_1, \ldots, b_n$ are real numbers. Prove that the roots of the polynomial equation

$$x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0,$$

are all real.

(Hint. If z ia a complex number, consider the geometrical meaning of a complex number w such that $|w - z| > |w - \bar{z}|$, where $\bar{z}$ is the conjugate of z.)

(via Ho Soo Thong)

*P7/76. Find the maximum area of a quadrilateral ABCD whose sides AB, BC, CD and DA are 25 cm, 8 cm, 13 cm, and 26 cm respectively.

(A.D.Villanueva)

*P8/76. Prove that the real roots of the polynomial equation

$$x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0,$$

where $a_1, \ldots, a_n$ are integers, are either irrational or integers.

(via C.T.Chong)

Hee Juay Guan and Tan Yee Lay have been awarded the prizes for correct solutions to P1/76 and P4/76 respectively.

Solutions to P1 - P4/76.

*P1/76. Let n be an integer greater than or equal to 2. Find integers $a_o, a_1, \ldots, a_{n-1}$ such that

$$1 + x + x^2 + \ldots + x^{n-1} = a_o + a_1(x-1) + a_2(x-1)^2 + \ldots + a_{n-1}(x-1)^{n-1}.$$

(H. N. Ng)

Solution by Hee Juay Guan

Multiplying the given equation by x-1, we have

$$x^n - 1 = a_o(x-1) + a_1(x-1)^2 + \ldots + a_{n-1}(x-1)^n.$$

Write $y = x - 1$:

$$a_o y + a_1 y^2 + \ldots + a_{n-1} y^n = (1+y)^n - 1 = \binom{n}{1} y + \binom{n}{2} y^{n-2} + \ldots + y^n.$$

where the $\binom{n}{r}$ are the binomial coefficients.

Hence $a_r = \binom{n}{r+1}$, $r = 0, 1, \ldots, n-1$.

Alternative solution by Chan Sing Chun.

Putting $x = 1$ in the given equation gives $a_o = n$. Differentiating with respect to x the given equation, we have

$$1 + 2x + 3x^2 + \ldots + (n-1) x^{n-2} = a_1 + 2a_2(x-1) + \ldots + (n-1)a_{n-1}(x-1)^{n-2}.$$

Putting $x = 1$, we obtain $a_1 = \frac{1}{2}n(n-1)$.

In general, if we differentiate the given equation r times and substitute $x = 1$, we have

$$r!a_r = 1.2...r + 2.3...(r+1) + 3.4...(r+2) + ... +$$
$$+ (n-r)(n-r+1)...(n-2)$$
$$= n(n-1)...(n-r)/(r+1)$$

Hence $a_r = \binom{n}{r+1}$.

Also solved by Proposer, Tay Yong Chiang, Ong Chai Seng, Lee Tong Heng, Lim Boon Tiong.

Two incomplete solutions were received.

P2/76. Taylor's theorem states: if $f^{(n-1)}(x)$ is continuous for $a \leq x \leq a + h$, and $f^{(n)}(x)$ exists for $a < x < a + h$, then there is a real number θ_n, $0 < \theta_n < 1$, such that

$$f(a + h) = f(a) + hf'(a) + \tfrac{1}{2}h^2 f''(a) + ... +$$
$$+ \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta_n h)$$

Suppose that $f''(x)$ is continuous for $a-h < x < a+h$ and $f''(a) \neq 0$. If θ is the number (which depends on h) such that

$$f(a+h) = f(a) + hf'(a + \theta h),$$

where $0 < \theta < 1$, prove that $\theta \to \frac{1}{2}$ as $h \to 0$.

Generalize the result.

(via Louis H.Y. Chen and Y.K. Leong)

Solution by Proposers.

Applying Taylor's theorem to the function $f'(a + \theta h)$, we have

$$f'(a + \theta h) = f'(a) + \theta h f''(a + \varphi \theta h)$$

for some φ such that $0 < \varphi < 1$.

Hence given equation becomes

$$f(a+h) = f(a) + hf'(a)\ \theta h^2 f''(a + \varphi \theta h).$$

Again by Taylor's theorem, we have

$$f(a+h) = f(a) + hf'(a) + \tfrac{1}{2}h^2 f''(a + \psi h)$$

where $0 < \psi < 1$.

Thus $\quad \theta f''(a + \varphi \theta h) = \tfrac{1}{2} f''(a + \psi h)$

Letting $h \to 0$, and since $f''(a) \neq 0$, we obtain that $\theta \to \tfrac{1}{2}$.

The result may be generalized to: if $f^{(n+1)}(x)$ is continuous for $a - h < x < a + h$, $f^{(n+1)}(a) \neq 0$ and θ_n is the number in the Taylor expansion of $f(a + h)$, then $\theta_n \to 1/(n+1)$ as $h \to 0$.

This problem occurs as an exercise in G.H. Hardy, *A course of pure mathematics*, Cambridge, 1960, p. 288; however, the condition that $f^{(n+1)}(a) \neq 0$ seems to have been left out.

*P3/76. Let $a_0 = 0$, $a_1 = 1$, and for every integer n, $a_{n+2} = a_{n+1} + a_n$. Prove that for integers m, n,

(i) $a_m a_n + a_{m+1} a_{n+1} = a_{m+n+1}$,

(ii) $a_n a_{n+2} - a_{n+1}^2 = (-1)^{n+1}$;

(Y. K. Leong)

Solution by Proposer.

(i) Let m be a fixed integer. The statement is obviously true for $n = 0$ and 1. Assume that the statement is true for integers $n \leq k$, where $k \geqslant 2$. Then:

$$a_m a_{k+1} + a_{m+1} a_{k+2} = a_m (a_k + a_{k-1}) + a_{m+1} (a_{k+1} + a_k)$$

$$= (a_m a_k + a_{m+1} a_{k-1}) + (a_m a_{k-1} + a_{m+1} a_k)$$

$$= a_{m+k+1} + a_{m+k} = a_{m+k+2}.$$

Thus statement is also true for $n=k+1$, and hence for $n = 0, 1, 2, \ldots$.

On the other hand, suppose that $k \leqslant 0$ and that the statement is true for integers $n \geqslant k$. Then

$$a_m a_{k-1} + a_{m+1} a_k = a_m (a_{k+1} - a_k) + a_{m+1} (a_{k+2} - a_{k+1})$$

$$= (a_m a_{k+1} + a_{m+1} a_{k+2}) - (a_m a_k + a_{m+1} a_{k+1})$$

$$= a_{m+k+2} - a_{m+k+1} = a_{m+k}.$$

Thus statement is also true for n = k-1 and hence for n = 0, -1, -2, ...

(ii) The statement is clearly true for n = 0. So suppose that it is true for n = k where $k \geqslant 0$. Then

$$a_{k+1}a_{k+3} - a_{k+2}^2 = a_{k+1}(a_{k+2} + a_{k+1}) - a_{k+2}^2$$

$$= (a_{k+1} - a_{k+2})\,a_{k+2} + a_{k+1}^2$$

$$= -a_k a_{k+2} + a_{k+1}^2 = (-1)^{k+2}.$$

Thus statement is also true for n = k+1 and hence for n=0,1,2, ...

Finally, let $k \leqslant 0$ and suppose that statement is true for n = k. Then

$$a_{k-1}a_{k+1} - a_k^2 = (a_{k+1} - a_k)\,a_{k+1} - a_k^2$$

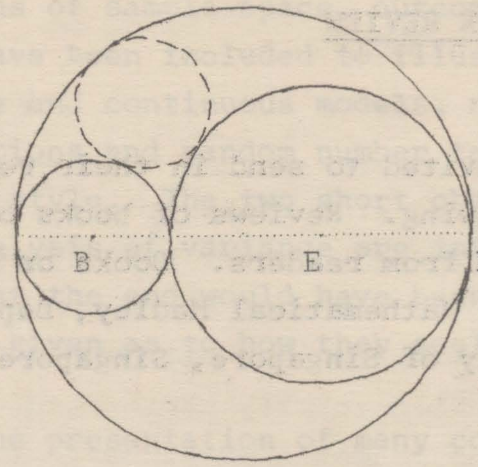
$$= a_{k+1}^2 - a_k(a_{k+1} + a_k)$$

$$= a_{k+1}^2 - a_k a_{k+2} = (-1)^k .$$

Thus statement is also true for n = k-1 and hence for n=0,-1,-2,...

Two incomplete solutions were received.

*P4/76. Find the radius of the circle tangential to the three given circles with centres B,E and diameter AF respectively, where AB = 1, EF = 2, BE = 3.



(A.P. Villanueva)

Solution by Tan Yee Lay.

Let C be the centre of the largest circle and D that of the required circle. Let r be the radius of the latter circle and θ = CED. Applying the cosine rule to $\triangle$s CED, BED respectively, we have

$$(3-r)^2 = 1 + (2+r)^2 - 2(2+r)\cos\theta,$$

$$(1+r)^2 = 3^2 + (2+r)^2 - 6(2+r)\cos\theta.$$

Solving for r, we get $r = 6/7$.

Also solved by Proposer, Chan Sing Chun, Chang Yew Kong, Tay Yong Chiang, Lim Boon Tiong.

One incorrect solution was received.

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