1976 INTER-SCHOOL MATHEMATICAL COMPETITION

The problems of the 1976 Inter-school Mathematical Competition are reproduced below. In Paper 1, each correct answer is marked with an asterisk. In Paper 2, outlines of the solutions to the problems are given.

PAPER 1.

Saturday, 19 June 1976 — 9.00 a.m. - 11.00 a.m.

1 The mean of the test scores of a group of 10 pupils was 62. Another group consisting of 40 pupils took the same test and the mean of the scores was 66. What is the mean of the scores of the two groups combined?

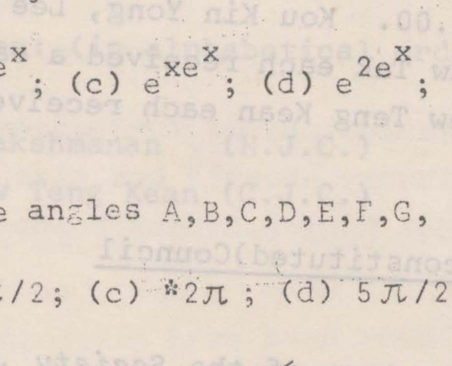
(a) 62.8; (b) 63.6; (c) 64; (d) 65.2; (e) * 78.5

2 Find dy/dx for $y = e^{e^x}$

(a) e^x; (b) e^{e^x}; (c) e^{xe^x}; (d) e^{2e^x}; (e) * $e^{(x+e^x)}$

3 The sum of the angles A,B,C,D,E,F,G, is equal to

(a) π; (b) $3\pi/2$; (c) *2π; (d) $5\pi/2$; (e) none of the above.



4 Let S be a set consisting of m elements. If f is a function from S onto S, then

(a) * f must be one-to-one; (b) f can never be one-to-one; (c) f may or may not be one-to-one, depending on S; (d) $m \geqslant 2$.

5 A rope of 7 cm. is used four times to form the perimeters of the following figures: a circle, a square, an equilateral triangle, and a rectangle of 1 cm. by 2.5 cm. Which figure gives the smallest area?

(a) circle; (b) square; (c) * equilateral triangle; (d) rectangle of 1 cm. by 2.5 cm.

6 A bag contains seven apples of which three are bad. If two apples are drawn randomly from the bag, what is the probability that one of them is good and the other is bad?

(a) 3/14; (b) 2/7; (c) 4/21; (d) 3/7; (e) * none of the above.

7 If the random variable X is uniformly distributed on the set of integers $\{1,3,5,\ldots,(2n-1)\}$, then the expected value of X is equal to

(a) n+1 ; (b) * n; (c) n-1; (d) $n+\frac{1}{2}$; (e) none of the above.

8 The equation $e^{2x} + be^{x} + c = 0$ has two real roots. We can conclude that one of the following is true:

(a) I only; (b) II only; (c) I and II only; (d) I and III only; (e) * I,II and III; where I: $b^2 - 4c > 0$, II: $b < 0$, III: $c > 0$.

9 Let x,y be real numbers such that $(x+iy)^2 = -2i$. Then one of the following holds.

(a) $x = -y$; (b) $x = y$; (c) $x = 1, y = 1$; (d) $x^2 = -1$; (e) * $x = 1, y = -1$.

10. Let n = 101 and

$$S = \{ a_x : a_x = e^{2\pi ix/n}, \quad x = 1,2,3,4,\ldots \},$$

where $i = \sqrt{-1}$ is the imaginary unit. The sum of all the (distinct) elements of S is

(a) * 0; (b) 1; (c) -1; (d) undefined; (e) 101.

11 Let A,B,C,D, be subsets of a set E. Assume that the number of elements of A,B,C,D,$A\cap B$,$A\cap C$,$A\cap D$,$B\cap C$,$B\cap D$,$C\cap D$,$A\cap B\cap C$, A B D, A C D, B C D and A B C D are 20, 25, 30, 35, 10, 11, 12,13,14,15,5,6,7,8,4 respectively. The number of elements of A B C D is

(a) 59; (b) * 57; (c) 82; (d) 60; (e) 51.

12 The sum $\sin\theta + \sin 3\theta + \ldots + \sin(2n-1)\theta$ is equal to

(a) $\dfrac{\sin 2n\theta}{2\cos\theta}$; (b) * $\dfrac{\sin^2 n\theta}{\sin\theta}$; (c) $n \sin\theta$; (d) $\sin^n\theta$;

(e) $\dfrac{\sin^3 n\theta}{\sin^2\theta}$.

13 Let A,B be points on two circles with centres P,Q, respectively. Assume that AB is a common tangent and the two circles touch at T. If $\widehat{ATP} = \theta$, then $\widehat{BTQ}$ is equal to

(a) $\frac{\pi}{4} + \theta$; (b) $\theta/2$; (c) $\frac{\pi}{2} + \theta$;

(d) $\pi - \theta$; (e) * $\frac{\pi}{2} - \theta$.

14 A cyclindrical bucket of uniform crossection is left out in the rain (falling vertically into it with uniform speed). The wind starts blowing uniformly across the falling rain. The rate at which the bucket is filled will

(a) increase; (b) * not be changed; (c) decrease;
(d) increase if the wind is light and decrease if the wind is strong; (e) increase if the rain is light and decrease if the rain is heavy.

15 Let A and B be two ports in two countries. Journey from A to B takes 12 days. Every mid-day a ship starts from A to B and another from B to A and they follow more or less the same course. The number of ships that each meets in the open sea is

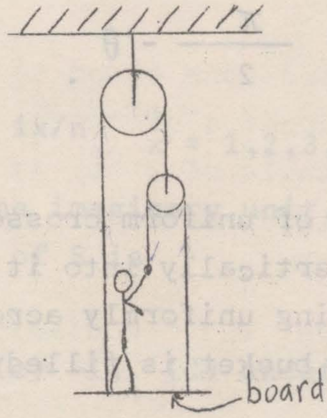
(a) 12; (b) * 23; (c) 24; (d) 13; (e) 25.

16 If a fast-moving car breaks suddenly, the front part of the car dips mainly because

(a) the friction on the front and rear wheels are different;
(b) the front part of the car is heavier than the rear part;
(c) of the impulsive nature of the friction;
(d) * the centre of gravity of the car is above the road;
(e) of both (a) and (b).

17 With what force must a man pull the rope to hold the board in position if the man weighs 60 kg? (Neglect the weight of the board, rope and pulley).

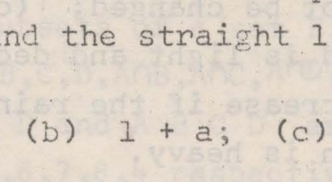
(a) * 15 kg; (b) 20 kg; (c) 30 kg; (d) 17.5 kg; (e) 12 kg.



18 The shortest distance between the parabola $y^2 = -4ax$ where $0 < a < 1$, and the straight line $x + y = 1$ is

(a) * $(1-a)/\sqrt{2}$; (b) $1 + a$; (c) $1 - a$; (d) $(1+a)/\sqrt{2}$; (e) $\sqrt{2} + a$.

19 On the square wire lattice shown below,



an ant goes from A to B, always crawling to the right or upwards. If all paths are equally likely, what is the chance that the ant passes through C?

(a) 4/25; (b) 3!/5! ; (c) 1/4 ; (d) 1/2 ; (e) * none of the above.

20 The first three prime numbers are

(a) o,1,2; (b) 1,2,3; (c) * 2,3,5; (d) 1,3,7; (e) -1,0,1.

21 Let y = f(x) be a polynomial function of degree 5, that is,

$$f(x) = a_o x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 .$$

Then its curve has

(a) 2 maxima and 2 minima; (b) at least one maximum and one minimum; (c) no maximum or minimum; (d) 3 points of inflection; (e) * at least one point of inflection.

22 How may zeros are there at the end of the number 20! ?

(a) 2; (b) * 4; (c) 6; (d) 10; (e) none of the above.

max no of

23 How many points of intersection can three circles and three straight lines have?

(a) 18; (b) 24; (c) 26; (d) * 27; (e) none of the above.

24 A function from a set A to a set B is a rule which assigns to each element of A a unique element of B. What is the number of functions from A to B if A has m elements and B has n elements, where m and n are positive integers ?

(a) m; (b) n; (c) m^n; (d) * n^m ; (e) none of the above.

25 How many pairs of positive integers m, n are there such that $m^2 - n^2$ is positive and a divisor (factor) of 2 x 3 x 5 ?

(a) 1; (b) 3; (c) * 4; (d) 6; (e) none of the above.

26 Find the number of solutions of $\theta (0 < \theta < 2\pi)$ of the equation $\tan \theta = 2 \tan \frac{1}{2}\theta$.

(a) * 0; (b) 1; (c) 2; (d) 3; (e) none of the above.

27 A hill whose base is a circle of radius 1 km lies between two towns A and B such that AB passes through the centre of the base and AB = $2\sqrt{2}$ km. Find the length of the shortest path from A to B.

(a) 4 km; (b) $2(\sqrt{2}-1)+\pi$ km; (c) * $2 + \frac{1}{2}\pi$ km;
(d) $2\sqrt{2}$ km; (e) none of the above.

28 Find the number of even integers n for which $2^n + 1$ is divisible by 3.

(a) * 0; (b) 1; (c) 7; (d) infinite; (e) none of the above.

29 Let S be a sample space and A and B be two independent events with probabilities P(A) and P(B) which are both equal to ½. Which of the following statements is correct?

(a) A and B are mutually exclusive; (b) $A \cup B = S$; (c) A = B;
(d) * $P(A \cap B) = P(A^c \cap B^c)$ where A^c and B^c are the complements of A and B respectively; (e) $P(A \cup B) = \frac{1}{4}$.

30 The value of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^5$$

is

(a) 0; (b) ∞ ; (c) $(1/6)^5$; (d) π ; (e) * none of the above.

31 Let $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ -x^2 & \text{if } x \text{ is irrational.} \end{cases}$

Then

(a) f is <u>not</u> a function; (b) f is continuous everywhere; (c) f is discontinuous everywhere; (d) * f is differentiable at exactly one point; (e) none of the above.

32 Mr A and Mr B each tosses 5 fair coins. What is the probability that they obtain the same number of heads?

(a) 16/32; (b) $126/32^2$; (c) * $252/32^2$; (d) $152/32^2$; (e) none of the above.

33 The minimum number of multiplications and divisions involved in solving a general system of 5 linear equations with 5 unknowns by elimination method is approximately

(a) 55; (b) * 65; (c) 75; (d) 85; (e) 95.

34 Let n and t be two positive integers such that t is not a multiple of n. If $w = \cos(2\pi/n) + i\sin(2\pi/n)$, then the value of $1 + w^t + w^{2t} + \ldots + w^{(n-1)t}$ is equal to

(a) 1; (b) -1; (c) $-w^{nt}$;(d)* 0; (e) none of the above

35 The smallest positive integer which has 12 distinct positive factors (including 1) is

(a) 30; (b) 40; (c) 50; (d) 84; (e) * none of the above.

36 A die is loaded in such a way that the probability of one side showing up is 1/5 and the probabilities of the other five sides showing up are equal. Suppose the die is rolled three times and two 5's and one 6 are obtained. Using the maximum likelihood method, which side would you infer to have probability 1/5 of showing up?

(a) the side with face value 2; (b) the side with face value 3; (c) the side with face value 4; (d)*the side with face value 5; (e) the side with face value 6.

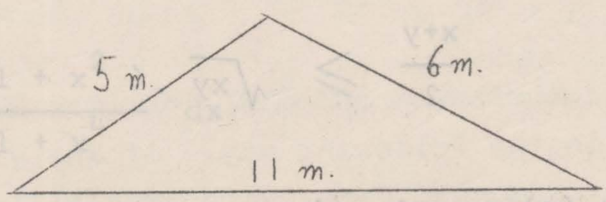
37 Let θ_1 and θ_2 be the angles between the positive x-axis and the lines $5x + 3y + 7 = 0$ and $6x = 10y - 7$ respectively. The angle $\theta_1 - \theta_2$ is

(a) 45 ; (b) 37° 30' ; (c) * 90° ; (d) 75° ;
(e) none of the above.

38 A and B together can do a piece of work in 24 days. If A can do only two-thirds as much as B, how long will it take A to do the work alone?

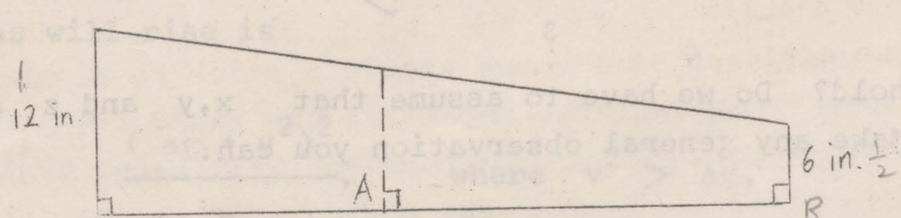
(a) * 60 days; (b) 36 days; (c) 28.8 days; (d) none of the above.

39 Mr A informed his friends one morning that a neighbour told him that he had been offered a piece of ground for a garden. He said it was triangular in shape, with the dimensions as shown in the diagram. As the price proposed was one dollar per square meter, what will the cost be?



(a) 15 dollars; (b) $\sqrt{182}$ dollars; (c) 22 dollars; (d) * none of the above.

40 A man had a board measuring 10 feet in length, 6 inches wide at one end, one 12 inches wide at the other, as shown in the illustration. How far from B must the straight cut at A be made in order to divide it into two pieces of equal area?



(a) $\frac{5\sqrt{13} - 5}{2}$; (b) * $5\sqrt{10} - 10$; (c) 6; (d) none of the above.

PAPER 2

Saturday, 19 June 1976 2.00 p.m.- 5.00 p.m.

1 Let ABC be an equilateral triangle and let P,Q,R, be points on BC, CA,AB respectively such that $\triangle$ PQR is equilateral.

(a) Prove that P,Q,R divide the sides BC, CA, AB in the same proportion. (b) Find the minimum area of $\triangle$PQR.

2 (a) Show that if $x \geqslant 0$ and $y \geqslant 0$, then

$$\frac{x+y}{2} \geqslant \sqrt{xy} \; .$$

(b) Use (a) or otherwise to show that if x,y,z and w are nonnegative, that is, $\geqslant 0$, then

$$\frac{x+y+z+w}{4} \geqslant \sqrt[4]{xyzw}$$

(c) Does the inequality

$$\frac{x+y+z}{3} \geqslant \sqrt[3]{xyz}$$

hold? Do we have to assume that x,y and z are nonnegative? Make any general observation you can.

3 During an archaelogical find, a huge prehistoric dictionary D was unearthed. All words in the dictionary were found to be juxtapositions of the two alphabets a and b (e.g. aaab, abbbba, etc). Words were ordered as is done in an ordinary English dictionary. Further investigations revealed that there was no end to the number of words in D. It is understood that a method was known to makers of the dictionary, whereby for a given set S of words in D, one could scan through all the words in S without missing out any. What do you think was the method?

4 Evaluate the following integrals, giving your answers to three significant figures in each case.

(i)
$$\int_1^2 \frac{(1 + x^2)}{1 + x^4}\, dx$$

(ii)
$$\int_{-3/2}^{3/2} \frac{x^3 \tan x}{e^{2x} - e^{-2x}}\, dx$$
↗ odd function.

5 A bicycle running without slipping along a level road with velocity v, throws off particles of mud from the rims of its wheels. If a is the radius of the wheel, show that the greatest height to which any of the mud particles will rise is

$$\frac{(ag + v^2)^2}{2gv^2}, \quad \text{where } v^2 > ag,$$

g being the acceleration due to gravity.

6 Prove that

$$1 + \tfrac{1}{2} + 1/3 + \ldots + 1/n ,$$

where $n \geqslant 2$, is not an integer.

7 Let $V(\alpha)$ denot the volume obtained by revolving about the x-axis the area bounded by the curve $y = x^6 - x^5 + x^4 - x^3 + x^2 - x - \alpha$, the x-axis, the y-axis and the line $x = 1$. Find the number α_0 such that $V(\alpha_0) \leq V(\alpha)$ for all α. (Note that you are not required to find the value of $V(\alpha_0)$, but you must show that $V(\alpha_0) \leq V(\alpha)$ for all α.)

8 (a) A spider crawls randomly along the sides of a square ABCD in the following way. At any vertex, it moves to an adjacent vertex with probability ½. If it starts from A, show that the expected number of steps it takes to reach C is 4.

(b) Now suppose the spider crawls randomly along the edges of a cube, and at any vertex it moves to an adjacent vertex with probability 1/3. What is the expected number of steps it takes to move from a vertex to the furthest vertex?

9 With what part of a sabre should a stick be slashed in order not to feel the impact on the hand? (The sabre may be assumed to be a uniform rod of length l and the handle is at one end of the rod).

10 Let $P_1, P_2, \ldots, P_r$ be r distinct primes. Let A_r be the set of all rational numbers of the form

$$\frac{a_1}{P_1} + \frac{a_2}{P_2} + \ldots + \frac{a_r}{P_r},$$

where $a_1, a_2, \ldots, a_r$ are integers. Find the smallest positive rational number in A_r.

If $P_1 = 5$, $P_2 = 7$, $P_3 = 13$, find integers a_1, a_2, a_3 such that

$$\frac{a_1}{5} + \frac{a_2}{7} + \frac{a_3}{13}$$

is the smallest positive rational number in A_3.

11 Show that $\alpha + \beta + \gamma = n$ (n a positive integer) if and only if $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$. Hence, or otherwise, prove that

$$x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) = 4xyz$$

if $yz + zx + xy = 1$.

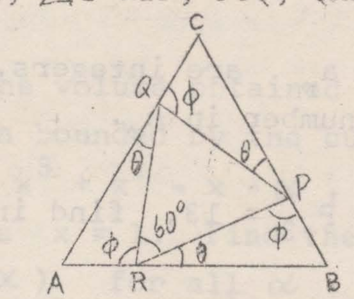
12 Let E be the set of all even integers, that is

$$E = \{0, \pm 2, \pm 4, \pm 6, \pm 8, \ldots\}$$

An integer p in E, greater than 1, is called a simple number in E if there is no d in E, $1 < d < p$, such that d is a factor of p.
Is the factorisation of every integer in E into simple numbers (in E) unique? (The different orderings of the same simple numbers are considered as the same factorisation.) If not, give an integer which can be factored in two different ways in E.

Solutions to Paper 2

1 (a) Let $\hat{BRP} = \theta$, $\hat{QRA} = \phi$. Then $\theta + \phi = 120^\circ$.
In $\triangle$ BRP, $\hat{BPR} = \phi$, Similarly for $\triangle$s PCQ and AQR.
Since RP = PQ = QR, $\triangle$s RBP, PCQ, QAR are congruent.
Hence the result.



(b) Let AB = a, AR = x,

Area of $\triangle PQR = \triangle = \frac{1}{2}a^2 \sin 60^\circ - \frac{3}{2}x(a-x) \sin 60^\circ$

$$= \frac{\sqrt{3}}{4}(a^2 - ax + x^2).$$

$$= \frac{\sqrt{3}}{4}\left\{(x - \tfrac{1}{2}a)^2 + \tfrac{3}{4}a^2\right\}$$

Therefore, $\triangle$ is minimum when $x = \frac{1}{2}a$,

and minimum area is $\frac{3\sqrt{3}}{16}a^2$.

2 (a) Starting from $(x - y)^2 \geqslant 0$, deduce

$$\frac{x + y}{2} \geqslant (xy)^{\frac{1}{2}} \quad \text{for } x \geqslant 0,\ y \geqslant 0.$$

(b) Writing $u = \frac{x + y}{2}$, $v = \frac{z + w}{2}$ and repeated use of 2(a) yield the result.

2 (c) In 2(b) let $w = \dfrac{x + y + z}{3}$;

then $\dfrac{x + y + z}{3} \geq (xyz)^{\frac{1}{4}} \left\{ \dfrac{x + y + z}{3} \right\}^{\frac{1}{4}}$,

which gives the result by taking $\dfrac{x + y + z}{3}$ to the l.h.s.

Yes, x,y,z must be non-negative otherwise the inequality will not hold. For example set x = y = -1, z =1.

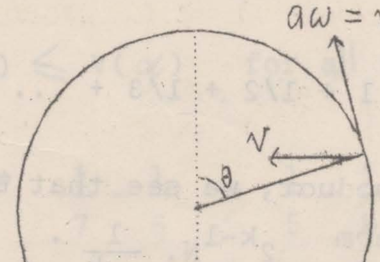
3 Given a set S, let S_n be the subset of S consisting of words of length n. One can then enumerate all the words in S_1 (there can at most be two words:a or b), and then enumerate the words in S_2 (there can at most be four words-- in general S_n can have at most 2^n words) and so on. In this manner no word in S will be missed out.

4 (i) Use the substitution y = x - 1/x. (Ans: 0.576)

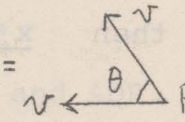
(ii) Let $f(x) = \dfrac{x^3 \tan x}{e^{2x} - e^{-2x}}$

then f(-x) = - f(x). Hence the value of the integral is zero.

5



5 Let w be the angular velocity of the wheel. Since the bicycle is running <u>without</u> slipping, the speed of M, the point of contact, is zero. Hence v = aw. Then the velocity of P relative to the ground =



Let $h(\theta)$ be the maximum height above the ground attained by a particle thrown off from P. Then

$$h(\theta) = a(1 + \cos\theta) + \frac{(v \sin\theta)^2}{2g}$$

$$= a + a\cos \quad + \frac{v^2}{2g}\left(1 - \cos^2\theta\right)$$

$$= \left(a + \frac{v^2}{2g}\right) + \frac{a^2 g}{2v^2} + \frac{v^2}{2g}\left(\cos\theta - \frac{ag}{v^2}\right)^2$$

$$= \frac{(ag + v^2)^2}{2gv^2} - \frac{v^2}{2g}\left(\cos\theta - \frac{ag}{v^2}\right)^2.$$

Thus the maximum height reached by one of the particles thrown from the rim

$$= \frac{(ag + v^2)^2}{2gv^2}.$$

6 Let 2^k $(k \geq 1)$ be the highest power of 2 not greater than n, and let N be the product of all the <u>odd</u> integers from 1 to n inclusive. Consider the product

$$2^{k-1} N (1 + 1/2 + 1/3 + \ldots + 1/n).$$

Expanding this product, we see that each term is an integer except for the term $2^{k-1}N.\frac{1}{2^k}$. Hence the result.

7 Let $f(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x$

Then $$V(\alpha) = \pi \int_0^1 \left(f(x) - \alpha \right)^2 dx$$

$$V(\alpha) = \pi \int_0^1 f^2(x)\,dx - 2\pi\alpha \int_0^1 f(x)\,dx + \pi\alpha^2$$

(1)... $$\frac{dV}{d\alpha} = -2\pi \int_0^1 f(x)\,dx + 2\pi\alpha$$

(2)... $$\frac{d^2V}{d\alpha^2} = 2\pi$$

From (1) $\frac{dV}{d\alpha} = 0$ if and only if $\alpha = \int_0^1 f(x)\,dx$

$$= \frac{1}{7} - \frac{1}{6} + \frac{1}{5} - \frac{1}{4} + \frac{1}{3} - \frac{1}{2}$$

$$= \alpha_0 .$$

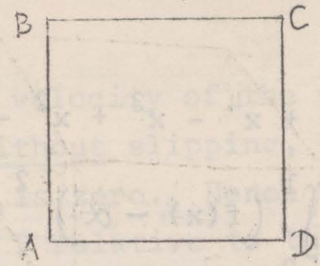
From (2) $\alpha = \alpha_0$ is a minimum.

Thus $V(\alpha_0) \leq V(\alpha)$ for all α.

where $$\alpha_0 = \frac{1}{7} - \frac{1}{6} + \frac{1}{5} - \frac{1}{4} + \frac{1}{3} - \frac{1}{2}$$

$$= -\frac{101}{420}$$

8(a)



Let E_A, E_B, E_D be the expected number of steps spider takes to reach C starting from A,B,D, respectively.

Then

(1) ... $E_B = E_D$ by symmetry.

(2) ... $E = (\frac{1}{2})(1 + E_B) + (\frac{1}{2})(1 + E_D)$

(3) ... $E_B = \frac{1}{2}(1) + \frac{1}{2}(1 + E_A)$.

In obtaining (2) and (3) we have used the fact that it take <u>one</u> step to go from A to B or A to D or B to C or B to A. from (1), (2), (3) we get $E_A = 4$.

<u>Alternatively</u>: Let P_n be the probability that it takes 2n steps to reach C starting from A. (Note that the number of steps must be <u>even</u>)

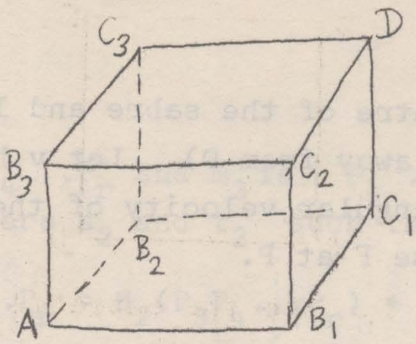
Then $\left.\begin{array}{l} P_1 = \frac{1}{2} \\ P_n = (\frac{1}{2}) P_{n-1} \end{array}\right\} \Rightarrow P_n = (\frac{1}{2})^n$.

Thus $E_A = \sum_{n=1}^{\infty} (2n) \cdot (\frac{1}{2})^n$

$= \sum_{n=1}^{\infty} n(\frac{1}{2})^{n-1}$

$= 4$.

8(b)



Let E_A, E_{B_i}, E_{C_2} be defined as in 8(a)

Then

(1) $E_{B_1} = E_{B_2} = E_{B_3}$ by symmetry.

$E_{C_1} = E_{C_2} = E_{C_3}$

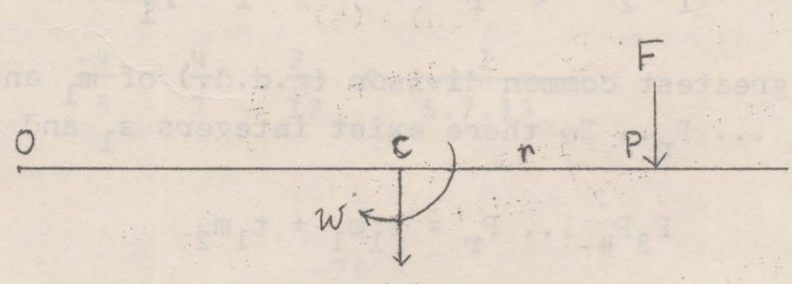
(2) $E_A = \sum_{i=1}^{3} (\frac{1}{3}) \quad (1 + E_{B_i})$

(3) $E_{B_1} = (\frac{1}{3})(1 + E_A) + \frac{1}{3}(1 + E_{C_1}) + \frac{1}{3}(1 + E_{C_2})$

(4) $E_{C_1} = (\frac{1}{3})(1) + \frac{1}{3}(1 + E_{B_1}) + \frac{1}{3}(1 + E_{B_2})$

Solving (1), (2), (3), (4) we get $E_A = 10$.

9



9 Let C be the centre of the sabre and let it strike at P where CP = r (away from O). Let v be the velocity of C and w be the angular velocity of the sabre immediately after the impulse F at P.

Then

using Impulse = change of linear momentum,

(1) ...... $F = mv$;

using moment of the Impulse = change of angular momentum,

(2) ...... $Fr = (\frac{1}{3})\ m\ (\frac{l}{2})^2\ w$

Now if the impact at O is to be zero, the velocity of O must be zero, thus

(3) ..... $0 = v - \frac{1}{2}l\ w$.

From (1), (2) and (3) we get $r = l/6$.

10 (1) The solution depends on the following result: if a and b are positvie integers with greatest common divisor d then there exist integers s and t such that

$$d = sa + tb.$$

Let $n = P_1\ P_2\ \dots\ P_r$ and let $m_i = \frac{n}{P_i}$

The greatest common divisor (g.c.d.) of m_1 and m_2 is $P_3P_4\ \dots\ P_r$. So there exist integers s_1 and t_1 such that

$$P_3P_4\ \dots\ P_r = s_1m_1 + t_1m_2$$

The g.c.d. of $P_3P_4\ldots P_r$ and m_3 is $P_4\ldots\ldots P_r$, so there exist integers s_2 and t_2 such that

$$P_4P_5\ldots P_r = s_2(P_3P_4\ldots P_r) + t_2\, m_3$$

$$= S_2 S\,(m_1) + S_2 t_1(m_2) + t_2(m_3)$$

Repeating the above argument we can find integers $a_1, a_2, \ldots a_r$ such that

$$1 = a_1m_1 + a_2m_2 + \ldots + a_rm_r$$

With these integers $a_1, \ldots, a_r$ we have

$$\frac{a_1}{P_1} + \frac{a_2}{P_2} + \ldots + \frac{a_r}{P_r} = \frac{a_1(m_1) + a_2(m_2) + \ldots + a_r(m_r)}{n}$$

$$= \frac{1}{n} = \frac{1}{P_1P_2\ldots P_r}$$

Then the smallest positive rational number in Λ_r is $\frac{1}{P_1P_2\ldots P_r}$

10(ii) By trial (or using the Euclidean algorthm) we get

$$\frac{1}{5} + \frac{-3}{7} + \frac{3}{13} = \frac{1}{5.7.13}$$

The answer is <u>not</u> unique; for example, we also have

$$\frac{-4}{5} + \frac{4}{7} + \frac{3}{13} = \frac{1}{5.7.13}$$

(a) If $\alpha + \beta + \gamma = n\pi$ ....(1)

then $\tan(\alpha + \beta) = -\tan\gamma$ .....(2) .

If follows that $\dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta} = -\tan\gamma$ ....(3)

or $\tan\alpha + \tan\beta = -(1 - \tan\alpha \tan\beta)\tan\gamma$ ...(4)

and so $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$ ....(5)

To prove the converse, it suffices to observe that the steps (1) – (5) can be retraced. Note that in (4), $1 - \tan\alpha\tan\beta$ cannot be zero, otherwise we would have

$$\tan\alpha\tan\beta = 1,$$

and

$$\tan\alpha + \tan\beta = 0 .$$

They would then imply $\sin^2\alpha + \cos^2\alpha = 0$, which is absurd.

(b) Let $x = \dfrac{1}{\tan\alpha}$, $y = \dfrac{1}{\tan\beta}$, $z = \dfrac{1}{\tan\gamma}$. Then $\sin 2\alpha = \dfrac{2x}{1+x^2}$

$\cos 2\alpha = \dfrac{1 - x^2}{1 + x^2}$ and similar expressions for $\sin 2\beta$, $\sin 2\gamma$

$\cos 2\beta$, $\cos 2\gamma$.

The condition $yz + zx + xy = 1$ is equivalent to

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$$

From (a), $\alpha + \beta + \gamma = n\pi$

or $$2\alpha + 2\beta + 2\gamma = 2n\pi$$

By expanding $\sin(2\alpha + 2\beta + 2\gamma) = \sin 2n\pi = 0$, we get

$$\sin 2\alpha \cos 2\beta \cos 2\gamma + \cos 2\alpha \sin 2\beta \cos 2\gamma$$

$$+ \cos 2\alpha \cos 2\beta \sin 2\gamma = \sin 2\alpha \sin 2\beta \sin 2\gamma .$$

Hence $x(1 - y^2)(1 - z^2) + (1 - x^2)y(1 - z^2)$

$$+ (1 - x^2)(1 - y^2)z = 4xyz$$

20 No. For example, 60 = 6 x 10

= 30 x 2 ,

and 2, 6, 10, 30 are simple sumbers in E.

* * * * * * * * *

'When a student makes really silly blunders or is exasperatingly slow, the trouble is almost always the same; he has no desire at all to solve the problem, even no desire to understand it properly, and so he has not understood it. Therefore, a teacher wishing seriously to help the student should, first of all, stir up his curiosity, give him some desire to solve the problem. The teacher should also allow some time to the student to make up his mind, to settle down to his task.

'Teaching to solve problems is education of the will. Solving problems which are not too easy for him, the student learns to persevere through unsuccess, to appreciate small advances, to wait for the essential idea, to concentrate with all his might when it appears. If the student had no opportunity in school to familiarize himself with the varying emotions of the struggle for the solution his mathematical education failed in the most vital point.'

George Polya (1887 -)