PROBLEMS AND SOLUTIONS

A book-voucher prize will be awarded to the best solution of a starred problem. Only solutions from Junior Members and received before 1 July 1976 will be considered for the prizes. If equally good solutions are received, the prize or prizes will be awarded to the solution or solutions sent with the earliest postmark. In the case of identical postmarks, the winning solution will be decided by ballot.

Problems or solutions should be sent to Dr. Y. K. Leong, Department of Mathematics, University of Singapore, Singapore 10. Whenever possible, please submit a problem together with its solution.

*P1/76. Let n be an integer greater than or equal to 2. Find integers $a_o, a_1, \ldots, a_{n-1}$ such that

$$1 + x + x^2 + \ldots + x^{n-1} = a_o + a_1(x-1) + a_2(x-1)^2 + \ldots + a_{n-1}(x-1)^{n-1}.$$

(H. N. Ng)

P2/76. Taylor's theorem states: if $f^{(n-1)}(x)$ is continuous for $a < x < a+h$, and $f^{(n)}(x)$ exists for $a < x < a+h$, then there is a real number Θ_n, $0 < \Theta_n < 1$, such that

$$f(a+h) = f(a) + hf'(a) + \tfrac{1}{2}h^2f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\Theta_n h).$$

Suppose that $f''(x)$ is continuous for $a-h < x < a+h$ and $f''(a) \neq 0$. If Θ is the number (which depends on h) such that

$$f(a+h) = f(a) + hf'(a + \Theta h),$$

where $0 < \Theta < 1$, prove that $\Theta \to \frac{1}{2}$ as $h \to 0$.

Generalize the result.

(via Louis H.Y. Chen and Y.K. Leong)

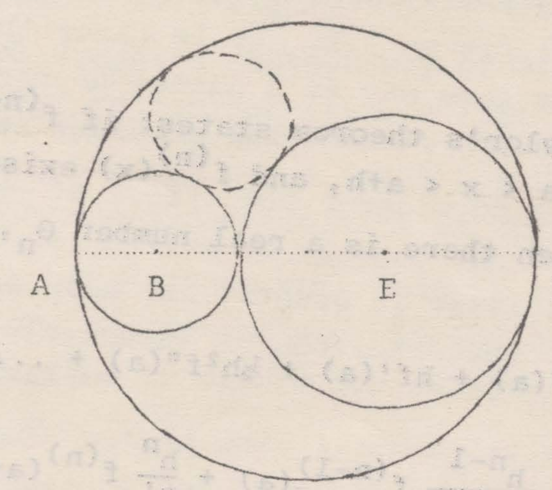
*P3/76. Let $a_o = 0$, $a_1 = 1$, and for every integer n, $a_{n+2} = a_{n+1} + a_n$. Prove that for integers m, n,

(i) $a_m a_n + a_{m+1} a_{n+1} = a_{m+n+1}$,

(ii) $a_n a_{n+2} - a_{n+1}^2 = (-1)^{n+1}$.

(Y. K. Leong)

*P4/76. Find the radius of the circle tangential to the three given circles with centres B, E and diameter AF respectively, where AB = 1, EF = 2, BE = 3.



(A.P. Villanveva)

Tay Yong Chiang has been awarded the prizes for correct solutions to P7/75 and P9/75.

Solutions to P7 and P9/75.

*P7/75. If $0 < x < \frac{1}{4}$, find the sum of the infinite series

$$1 + \binom{2}{1}x + \binom{4}{2}x^2 + \ldots + \binom{2r}{r}x^r + \ldots,$$

where $\binom{2r}{r}$ is the binomial coefficient $(2r)!/(r!)^2$.

(Chan Sing Chun)

Solution by Tay Yong Chiang.

The coefficient $\binom{2r}{r}$ is

$$\frac{2r(2r-1)(2r-2)\ldots 3.2.1}{r!\,r!}$$

$$= \frac{2^r(2r-1)(2r-3)\ldots 3.1}{r!}$$

$$= \frac{2^r 2^r (r-\frac{1}{2})(r-\frac{3}{2})\ldots\frac{3}{2}.\frac{1}{2}}{r!}$$

$$= \frac{(-1)^r 4^r (-\frac{1}{2})(-\frac{3}{2})\ldots(-\frac{1}{2}-r+2)(-\frac{1}{2}-r+1)}{r!}$$

which is the coefficient of x^r in the expansion of

$(1-4x)^{-\frac{1}{2}}$. Hence the series is equal to $(1-4x)^{-\frac{1}{2}}$ if $|x| < \frac{1}{4}$.

*P9/75. Show that the following system of equations cannot have real solutions w, x, y, z:

$$x^2 + yz = 0$$

$$w^2 + yz = 0,$$

$$xy + yw = 1,$$

$$xz + zw = 1.$$

(H. N. Ng)

Solution by Tay Yong Chiang.

The given equations may be written in the matrix form

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Taking determinants, we have

$$(xw - yz)^2 = -1 .$$

Hence x, y, z, w cannot be all real.

No solutions to P8 have been received. Those interested in the solution of P10/75 should write to Dr. H. N. Ng, Department of Mathematics, University of Singapore, Singapore 10.