NOTES ON MATHEMATICIANS

4. Henri Poincaré (1854-1912)

C. T. Chong
University of Singapore

For over two hundred and fifty years, the French school of mathematics had exerted great influence on the development of mathematical science. From the time of Descartes [1], Fermat [2], Pascal [3], Lagrange [4] and Laplace [5], the contributions by the French on geometry, arithmetic, probability theory, celestial and analytical mechanics had been impressive. Their achievements, however, were overshadowed by those of the German mathematician Gauss [6], whose universal genius remains unrivaled till this day — after almost two centuries. Yet in adhering to its great tradition of producing men of arts, sciences and literature, two French mathematicians did emerge and make their marks on the book of mathematical luminaries during the latter half of Gauss' era. They were Cauchy [7] and Galois [8]. While it is possible to conclude that Cauchy's work represented the best that he could have contributed to knowledge during his lifetime of sixty-eight years, one would expect Galois to have done more had he not ended his life rather abruptly at the early age of twenty-one. Assuming, as is generally accepted, that the mind is most creative before forty, there were still nearly twenty years lying ahead for Galois. From this point of view it is perhaps rather fortunate that the important role played by French school did not end suddenly when Galois had died and Cauchy had grown old. The man who came to succeed them did more than just keep up the tradition. He gained the unofficial title "*the greatest living mathematician*" from the Germans (justly endowed on Gauss) and made it a French property. He was Jules-Henri Poincaré.

There is as yet no published book in English devoted to the life and work of Poincaré. Nevertheless, he was accorded a chapter in E.T.Bell's *Men of Mathematics* [9], and his biographies can be found in both the *Obituaries of Fellows of the Royal Society (1915)* and the *Bulletin of the American Mathematical Society (1915)* (as a supplement to E. Lebon's *Savants du Jour:Henri Poincaré, Biographie, Bibliographie analytique des Ecrits* (Paris, 1912)). There are, however, numerous French memoirs on Poincaré, and the one written by the famous French mathematician Hadamard [10] (somewhat younger than Poincaré), entitled *"Poincaré le mathematicien"* (published in *Revue de métaphysique et de morale (1913)*), is of unsurpassed excellence. The memoir *Eloge Historique d'Henri Poincaré* (Paris, 1913) written by Poincaré's contemporary Gaston Darboux [11] is also a very good biographical source.

The life. "Henri Poincaré was a French savant who looked alarmingly like the popular image of a French savant. He was short and plump, carried an enormous head set off by a thick spade beard and splendid mustache, was myopic, stooped, pince-nez glasses attached to a black silk ribbon." [12]. This was probably the impression that one had of Poincaré when the man was at the pinnacle of his career. He was born on April 29, 1854 at Nancy, Francy, the son of a physician. It seems that the Poincaré's had long been an established family at Nancy. Poincaré's grandfather, for example, was attached to the military hospital at the age of twenty. His uncle, Antoine, became the inspector-general of The department of roads and bridges. The younger generation appeared to have been most distinguished: Henri rose to become the greatest mathematician of his time, and Raymond, his cousin (son of Antoine), became the President of France during the first world war.

According to Darboux, Poincaré passed his childhood in a circle of "savants, university men and polytechnicians." His father was a professor in the medical faculty at the university, and his mother was a very gifted woman who paid special attention to the education of her two children

(Poincaré and his sister).

As a child he had shown great talent in reading. His ability to recall from memory every detail of the books that he had read fascinated the teachers. In connection with this, it is worthwhile to note that the "mathematicizing" of Poincaré was markedly different from that of others — whereas most mathematicians understand mathematical theorems and proofs with the aid of paper and pencil, Poincaré did it only with his ears. Due to his poor eye-sight, he attended all mathematical lectures in the university by listening, without taking a note or following the syllabus. People were amazed, but it seemed very natural to him.

His physical co-ordination had been bad since young. And at the age of five, he suffered an attack of diphtheria which left him with a paralyzed larynx for nine months. Perhaps it was because of the physical handicaps that more time was spent in reading than in playing with boys of his age.

He was not drawn towards mathematics until the age of fourteen or fifteen. (As a digression, the Swiss mathematical journal *l'Enseignement Mathematique* published in 1908 a report on the questionaire sent out to living distinguished mathematicians all over the world. One of the questions asked: In your recollection, at what period and under what circumstances did the taste for mathematics take possession of you? Ninety-three replies to this question were received. Thirty-five placed the period before ten years of age, forty-three from eleven to fifteen years of age, eleven from sixteen to eighteen years, three from nineteen to twenty years, and only one at twenty-six years of age.) Thus strictly speaking, Poincaré did not even belong to the most precocious group. His first love was natural history. Till the end of his life, the love for natural history and philosophy never ceased, and it explains why towards his last years he wrote so prolifically on the philosophy of science and mathematics. Yet his tremendous power in mathematical reasoning was beginning to show during adolescence: when one asked him to solve a difficult mathematical problem, "the reply came like an arrow."

He first attended the Nancy lycee [13] , but the education was disrupted due to the Franco-Prussian war in 1870-1871, during which time the Germans occupied Nancy. In his eagerness to read the enemy's newspaper, he taught himself German (he was then about sixteen years old). At the end of 1871 he passed the entrance examination of the Ecole Polytechnique [14] holding first place, despite the zero mark given to the undecipherable geometrical diagrams that he drew.

Several legends of his early days should be mentioned here if only for the sake of understanding the man better (see [9]). On one occasion some students wanted to test him out, eager to show that he was really not as good as what others had claimed. A tough mathematics question was put forth. The students gathered around and waited for the moment of triumph. Without seemingly giving any thought to the problem, Poincaré solved it on the spot and walked off, leaving behind the startled students asking, "How did he do it?" — A question that others were to repeat for many years to come.

His inability to draw pictures that resembled anything was well-known to his fellow students. Perhaps as a sign of their affection for him, an arts exhibition was held at the end of the year, featuring a collection of his great art works. To enhance appreciation, titles were given in Greek to each piece of work — "Horse", etc. Of course, not always accurately. If Poincaré had tried harder, he would probably have made a great artist in abstract painting!

In 1875 he left the Polytechnique (at the age of twenty one) and entered the School of Mines "with the intention of becoming an engineer." The course work there left him with some time to do mathematics. This he spent in working on a general problem in the theory of differential equations. He graduated from the School of Mines on March 26, 1879. In the same year, on August 1, he received the degree of doctor of mathematical sciences from the University of Paris. The thesis which he submitted originated from the results that he had obtained over the past three years. Darboux was asked to examine the work. "At the first glance, it was clear to me that the thesis was out of the ordinary and amply merited acceptance. Certainly it contained results enough to supply material for several good theses." He went on to say, however, that there were several places in the thesis where ideas should be elaborated, explanations be given, and errors be corrected. Poincaré obliged by taking up the suggestions, "but he explained to me that he had many other ideas in his head; he was already occupied with some of the great problems whose solution he was to give us."

On December 1, 1879, he was appointed to teach in the Faculté des Sciences de Caen. Two years later he was appointed *Maitre de Conferences* in mathematical analysis at the Université de Paris, and in 1886 (at the age of thirty two) he was promoted to the chair of professor of mathematical physics and the calculus of probabilities. He stayed on at the University of Paris till the end of his life. He did, however, make short visits to European countries and to the United States. For example, he was an invited lecturer at the 1904 International Congress of Arts and Sciences held in St. Louis, U.S.A., and in 1909 he was at the University of Göttingen, Germany, at the invitation of "the other great mathematician" David Hilbert [15].

The two had first met in the early spring of 1886 when Hilbert, as a young and aspiring mathematician of twenty-six, visited Paris. Poincaré was only six years older than Hilbert, but had already published more than a hundren papers. The impression that Poincaré gave Hilbert was not too overwhelming: "He lectures very clearly and to my way of thinking very understandably although, as a French student here remarks, a little to fast. He gives the impression of being very youthful and a bit nervous. Even after our introduction, he does not seem to be very friendly; but I am inclined to attribute this to his apparent shyness, which we have not yet been in a position to overcome because of our lack of linguistic ability." [16] It was not until the turn of the century that his fame began to catch up with Poincaré's.

In May of the year that the two men first met, Poincaré was elected to l'Académie des Sciences. The proposal for his election was accompanied by the statement that his work "is above ordinary praise and reminds us inevitably of what Jacobi [17] wrote of Abel [18] — that he had settled questions which, before him, were unimagined. It must indeed be recognized that we are witnessing a mathematical revolution comparable in every way to that which manifested itself, half a century ago, by the accession of elliptic functions."

If Hilbert was not overwhelmed in Paris by Poincaré's presence, the story related by the English mathematician Sylvester [19] tells us how others had felt on seeing the man for the first time. In 1885, the year before Hilbert visited Paris, Sylvester in his old age made a pilgrimage to the French capital to meet the man that the mathematical world had been talking about. He was astonished to discover that the heralded "new star in mathematics" was "a mere boy, so blond, so young." "In the presence of that mighty pent-up intellectual force my tongue at first refused its office, and it was not until I had taken some time (it may be two or three minutes) to peruse and absorb, as it were, the idea of his external youthful lineaments that I found myself in a condition to speak." [9] .

The mathematical contributions of Poincaré encompassed many branches of pure and applied mathematics: the theory of automorphic functions in analysis (perhaps his best work in pure mathematics), the theory of numbers, topology, the many-body problem in mathematical astronomy (the work for which he was awarded a prize by King Oscar II of Sweden in 1887 and made a Knight of the Legion of Honor by the French government), and mathematical physics. By the turn of the century, he took up several new interests: the philosophy of science and mathematics, and the psychology of mathematical invention. During the thirty-four years of his scientific career, he published more than thirty books on mathematical physics and astronomy and nearly five hundred research papers on mathematics. In addition to these he wrote two books of popular essays and three famous volumes on the philosophy of science and mathematics -- *Science and Hypothesis*, *The Value of Science*, *Science and Method*. Story has it that in the early days when these books were first published, it was not uncommon to see working men and women reading then during lunch or coffee hours in the side walk cafés of Paris. As Jourdain said, "One of the many reasons that he will live is because he made it possible for us to understand him as well as to admire him." [20] For the literary quality of his essays, he was elected in 1908 as one of the forty members of the prestigious French Academy. This is all the more remarkable since Poincaré was, first and foremost, a scientist.

The latter part of Poincaré's life was crowned with medals and honors. The gold medal of the Royal Astronomical Society, the Sylvester medal of the Royal Society, the Lobachevskii medal of the Physico-Mathematical Society of Kazan, and the Bolyai prize of the Hungarian Academy of Sciences [21] , had all been awarded to him on various occasions in recognition of his achievements in mathematics.

He was happily married with a son and three daughters, and loved classical music. His health had been good until he was taken ill by the enlargement of the prostate in 1908 while attending the International Congress Mathematicians at Rome, Italy. The trouble was temporarily relieved by surgeons and he resumed his work immediately upon returning to Paris. His health deteriorated and by December 1911 he began to feel that time was running out. He asked the editor of a mathematical journal to accept an unfinished paper on a problem which he believed to be very significant: "At my age, I may not be able to solve it, and the results obtained, susceptible of putting researchers on a new and unexpected path, seem to me too full of promise, in spite of the deceptions they have caused me, that I should resign myself to sacrificing them." (See [9]).

In an address on Poincaré's work, the mathematician Vito Volterra said in connection with the unfinished paper that "among the various ways of conceiving man's affection for life, there is one in which that desire has a majestic aspect. It is quite different from the way one usually regards the feeling of the fear of death. There come moments when the mind of a scientist engenders new ideas. He sees their fruitfulness and utility, but he knows that they are still so vague that he must go through a long process of analysis to develop them before the public will be able to understand and appreciate them at their just value. If he believes then that death may suddenly annihilate this whole world of great thoughts, and that perhaps ages may go by before another discovers them, we can understand that a sudden desire to live must seize him, and the joy of his work must be confounded with the fear of having it stop forever." [22] The problem was solved soon after the unfinished paper appeared, by the American mathematician George David Birkhoff [23] .

In the spring of 1912 he fell ill again. During that summer — in his fifty-ninth year — Poincaré died.

Mathematical achievements. It was Poincaré's style to change the topics of his lectures every year. They covered both pure and applied mathematics: automorphic function, topology (called *analysis situs* by him), number theory, the equilibrium of fluid masses, the mathematics of electricity, astronomy, thermodynamics and light, and the calculus of probability. During his visit to Göttingen in 1904, he lectured on integral equations. Many of these lectures were published soon after they had been delivered at the university.

We will only describe briefly Poincaré's mathematical achievements.

Pure mathematics: Before the age of thirty, he initiated the study of automorphic functions — functions that are invariant under a group of transformations of the form f(z) = (az+b)/(cz+d) where z is a complex variable, and a,b,c,d are complex numbers. He showed, for example, that one could use these functions to solve linear differential equations with rational algebraic coefficients. The theory of automorphic functions and forms has today developed into one of the deepest subjects in mathematics, closely related to the theories of complex multiplication and arithmetic (cf. [24] for details).

His *Analysis situs* (published in 1895) gave an early systematic treatment of topology. The notions of fundamental group and homology were first introduced there. The famous Poincaré conjecture in topology, despite the great advance made by Smale [25], remains unsolved till this day: Every compact, simply-connected three-dimensional manifold is homeomorphic to the three-sphere S^3.

In arithmetic, Poincaré showed that the notion of binary quadratic forms introduced by Gauss could be cast in geometric form. We note here that there are essentially two 'types' of mathematians: the ones who prefer to "mathematicize" *analytically* — resorting to geometric pictures only occasionally, and the ones who would like to put every mathematical problem into a *geometric* perspective. Poincaré belonged to the latter (even though he could not draw anything properly!)

Applied Mathematics: Poincaré's most outstanding contribution in this area was to the problem of 'three bodies'. The classical 'n-body problem' states: assuming that the masses, motions and relative distances of n-bodies (such as the sun, moon, and earth for n=3) are known at a given time, what will the configuration of the system be at any future time, if the n-bodies attract one another according to Newton's law of gravitation? Poincaré developed powerful new mathematical techniques to solve the three-body problem, and in the course of solving it, made fundamental discoveries on the behaviour of the solutions of differential equations near singularities. Commenting on the work, Weierstrass [26] said, "Its publication will inaugurate a new era in the history of celestial mechanics."

He contributed to the study of the evolution of celestial bodies by describing the conditions of stability of the 'pear-shape' figures. He showed that a spherical mass which rotates with increasing speed will deform into a 'pear-shaped' mass, then "hollow out more and more at its 'waist', until finally it separates into two distinct and unequal bodies" (see [9]).

In a paper on the dynamics of the electron published in 1906, he obtained — independently — many of the results of Einstein's special theory of relativity. And finally, in his extensive writings on probability theory he anticipated the concept of ergodicity that is basic to statistical mechanics.

Philosophy of science and mathematics. Poincaré's views on scientific hypotheses properly classifies him as a pragmatist. Josiah Royce [27] writes in the introduction to the English translation of Poincaré's *Foundations of*

Science [28] that essentially there are two kinds of useful hypotheses in science: those which are valuable precisely because they are verifiable or refutable through experiments, and those who are suggested by experience but are neither verifiable nor refutable. The importance of the first kind is evident, while Poincaré made explicitly prominent the existence and importance of the second kind of hypotheses. According to Poincaré, this second kind constitutes an essential human way of viewing nature. In the words of Royce," an interpretation rather than a portrayal or a prediction of the objective facts of nature, an adjustment of our conceptions of things to the internal needs of our intelligence, rather than a grasping of things as they are in themselves." This would imply that hypotheses of the second kind are adopted for convenience, by convention and because of the usefulness they give in interpreting physical phenomena. They are neither wholly subjective and arbitrary, nor are they imposed upon us by the necessity of physical reality. As a result, such hypotheses may be changed, amended or discarded in the course of time.

Poincaré gave examples to illustrate his point: "Masses are coefficients it is convenient to introduce into calculations. We could reconstruct all mechanics by attributing different values to all the masses. This new mechanics would not be in contradiction either with experience or with the general principles of dynamics. Only the equations of this new mechanics would be less simple." [29] "... Of two watches, we have no right to say that one goes true, the other wrong: we can only say that it is advantageous to conform to the indications of the first." [30] "Absolute space, that is to say, the mark to which it would be necessary to refer the earth to know whether it really moves, has no objective existence... The two propositions: "The earth turns around" and 'it is more convenient to suppose the earth turns around' have the same meaning; there is nothing more in the one than in the other." [30] Thus the Kantian notion [31] of *synthetic a priori* 'spatial geometry' made no sense to Poincaré. He believed that the geometry adopted by us was of the most *convenient* and *advantageous* kind; it was, however, no "truer" than any other kind of geometry.

Poincaré's philosophy of mathematics centered around two points: it is not possible to reduce all of mathematics to logic (by this he meant logic of propositional calculus together with an appropriate set of axioms which does not include the axiom of induction-for instance Hilbert's axioms for geometry), and there does not exist 'actual infinite' as proposed by Georg Cantor [32]. His argument was rather forceful in those days — when the foundations of set theory were yet to be laid by Zermelo [33], Fraenkel [34], Gödel [35], and von Neumann [36], whose works would not be completed until almost thirty years after his death, and also when the notion of 'synatical proof' was still in its embryonic stage. The logicist school contended that given the basic rules of logical reasoning and a set of axioms, one could derive from these all the true theorems. Poincaré, however, thought that such a feat could not be accomplished without the aid of mathematical intuition — something that could not be formalized into symbols. Of course, Russell [37] and Whitehead [38] were the chief proponents of logicism, while Hilbert held a somewhat similar view. Subsequent events showed that both sides had been partly wrong and partly right. Gödel in 1931 (at 24) showed that the logicists' ambition could be carried out in the case of propositional calculus but not in the case of the Peano [39] axioms for arithmetic. In a way, Gödel's results showed how seemingly eloquent philosophical arguments would collapse in the face of critical mathematical investigations. This was, of course, not the first time in the history of mankind. One need only recall Kant's 'absoluteness' of Euclidean geometry and the advent of non-Euclidean geometry that subsequently took place.

In set theory, Poincaré thought that Cantor's notion of 'actual infinity' was untenable as it led to a host of, by now classical, antinomies [40] . While various attempts were made to remedy the defects, Poincaré insisted that no such attempts would be successful unless it be proved that the resulting axioms were free from contradiction. This should be the case even when the truth of the axioms was intuitively clear. Little did he envisage that what he demanded was precisely what Gödel proved to be impossible twenty years after his death: if set theory is contradiction-free, no proof of this fact could be carried out (within the realm of the axioms).

It is perhaps regrettable that Poincaré with his universal genius could not see the significance of Cantor's work hidden underneath the apparently unremovable antinomies. Supporters of Cantor expressed their stand in different manners. Hermite [41] and Hadamard did it in a very mild way, while Hilbert was more outspoken. "Cantor's set theory is the finest product of mathematical genius, and one of the supreme achievements of purely intellectual human activity," he said. "And no one shall take away the paradise that Cantor has created fro us!" (see [16])

Psychology of mathematical invention. How is mathematical invention possible? This mental activity which seems to take least from the outside world, and which seems to "act only of itself and on itself" [42] , appears to be rather mysterious. Poincaré was probably the first mathematician to have analysed this activity and to have written out his findings in a language intelligible to the layman.

Ideally, a good mathematician should have a memory of a first-rate chess player (who can visualize and memorize various combinations of moves) and should be as accurate as a computer in calculations. This sometimes happens — as in the case of Gauss. But Poincaré admitted, for example, that he was a poor chess player and often made errors in calculations. How is it, then that he could produce mathematical works of the first order?

In his opinion, there is a certain feeling — the intuition of mathematical order — which is possessed by every good mathematician. It is a feeling which is hard to define, but with which one discovers harmonies in mathematical structures and relates various mathematical ideas. Most people will not have this delicate feeling, nor will they have the strength of memory and calculating ability that is beyond the ordinary. These people will be absolutely incapable of understanding higher mathematics. Some others will have this feeling in a slight degree, but will be very good in memory and calculation. They can understand mathematics and make applications, but they cannot create. Still others will possess this special intuition to a larger degree. Their memory and calculating ability may be just ordinary, but they can create good mathematics.

What then is mathematical invention? It is the ability to discern and choose, from among the infinite possibilities of combinations of mathematical entities, those which are useful and will lead to a new mathematical law. These 'correct' combinations reveal a hitherto unsuspected kinship between mathematical truths.

To further illustrate his point, Poincaré presented several instances from his own experiences. His account was so complete and excellent that one cannot but be tempted to reproduce it. The following is an excerpt from [42] .

> "For fifteen days I strove to prove that there could not be any functions like those I have since called Fuchsian functions [43] . I was then very ignorant; every day I seated myself at my work table stayed an hour or two, tried a great number of

combinations and reached no results. One evening, contrary to my custom, I drank black coffee and could not sleep. Ideas rose in crowds; I felt them collide until pairs interlocked, so to speak, making a stable combination. By the next morning I had established the existence of a class of Fuchsian functions, those which come from hypergeometric series; I had only to write out the results, which took but a few hours.

"Just at this time I left Caen, where I was then living, to go on a geological excursion under the auspices of the school of mines. The changes of travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the Fuchsian functions were identical with those of non-Euclidean geometry. I did not verify the idea;... but I felt a perfect certainty. On my return to Caen, I verified the result at my leisure.

"Then I returned my attention to the study of some arithmetic questions apparently without much success and without a suspicion of any connection with my preceeding researches. Disgusted with my failure, I went to spend a few days at the seaside, and thought of something else. One morning, walking on the bluff, the idea came to me, with just the same characteristic of brevity, suddenness and immediate certainty, that the arithmetic transformations of intermediate ternary quadratic forms were identical with those of non-Euclidean geometry.

"Thereupon I left for Mont-Velerien, where I was to go through my military service; so I was very differently occupied. One day, going along the street, the solution of the difficulty which had stopped me suddenly appeared to me. I did not try to go deep into it immediately, and only after my service did I again take up the question. I had all the elements and had only to arrange them and put them together. So I wrote out my final memoir at a single stroke and without difficulty."

Thus to Poincaré, the process of mathematical invention consists of three stages: conscious work on a difficult problem (work which will end up in failure due to lack of progress); unconscious work carried out by the sublimal self (the stage in which sudden illumination appears); and again conscious work (the stage of rearranging ideas, putting them in the right order). The success of the unconscious work cannot happen without the long, arduous and fruitless hours spent working consciously on the problem. The second period of conscious work is used to organise and relate the inspired ideas, and to verify their validity.

E.T. Bell called Poincaré "the last universalist." Judging from the tremendous amount of work carried out by Hilbert, it may be more appropriate to award the title to the man six years junior to Poincare, for no one ever since has been able to achieve as much. Yet Poincaré was beyond doubt a universalist of the very first class. He was probably the only scientist whose popular essays had been translated into no less than six foreign languages: English, German, Russian, Italian, Spanish and Japanese. He is the only mathematician whose name appears in the title of a mathematical journal: *Annales de l'institut Henri Poincaré*, published in two sections (mathematical physics and calculus of probabilities) by the French Academy of Science). His extensive writings were compiled and published by the Paris Academy of Science as *Oeuvres de Henri Poincaré (Collected works of Henri Poincaré)*, a total of ten volumes which appeared progressively from 1916 to 1954. His genius will be recognized in the future as it has always been, and it is unlikely that future generations will revise the judgement that he ranks among the greatest mathematicians of all time.

Notes

[1] René Descartes (1596-1650), philosopher and mathematician, Invented analytic geometry.

[2] Pierre Fermat (1601-1665), jurist and mathematician; contributed to the theory of arithmetic and probability; best known for his Last Theorem (see H.N. Ng, "An elementary problem whose solution can lead to fame: Fermat's Last Theorem," this *Medley*, vol.3, No.2 (1975), pp.54-61).

[3] Blaise Pascal (1623-1662), theologian and mathematician; contributed to geometry and founded, with Fermat, the mathematical theory of probability.

[4] Joseph-Louis Lagrange (1736-1813), contributed to arithmetic and celestial mechanics.

[5] Marquis Pierre Simon de Laplace (1749-1827), politician and mathematician- contributed to celestial mechanics, mathematical physics and probability.

[6] Carl Friedrich Gauss (1777-1855), 'the prince of mathematicians' (see [9]), contributed significantly to almost every branch of pure and applied mathematics. See C.T.Chong, "Notes on mathematicians 1: Carl

Friedrich Gauss," this *Medley*, Vol.3, No.1, (1975) pp. 6-10.

[7] Augustin Cauchy (1789-1857), contributed to analysis and was one of the founders of complex function theory.

[8] Evariste Galois (1811-1832), made fundamental discoveries on the solvability of equations by radicals at the age of seventeen; killed in a duel at twenty-one.

[9] E.T. Bell, *Men of mathematics*, Simon and Schuster, New York, 1937.

[10] Jacques Hadamard (1865-1963), French mathematician whom Birkhoff (see [23]) hailed as 'Poincare's successor'; contributed to number theory, analysis and geometry as well as educational, philosophical and psychological aspects of mathematics; taught at College de France; died at 98.

[11] Gaston Darboux (1842-1917), French mathematician; contributed to geometry.

[12] James Newmann, *The World of Mathematics*, Vol.2. *'Commentary on Poincaré,'* George Allen and Unwin, London, 1956.

[13] State maintained Secondary School at Nancy.

[14] Engineering School under the direction of the Ministry of the Armed Forces; established in 1794; has faculties of mathematics, mechanical engineering, physics, and chemistry.

[15] David Hilbert (1961-1943), German mathematician and theoretical physicist- taught at Göttingen; contributed to various branches of pure and applied mathematics; was known together with Poincaré as the two leading mathematicians in the early twentieth century; see [16] .

[16] Constance Reid, *Hilbert*, Springer Verlag, Berlin, 1970.

[17] Carl Jacobi (1804-1851), German mathematician; contributed to elliptic and abelian functions, arithmetic, dynamics, algebra.

[18] Niels Henrik Abel (1802-1829), Norwegian mathematician; contributed to abelian function theory. See Oystein Ore, *Niels Henrik Abel: Mathematician Extraordinary*, University of Minnesota Press, Minneapolis, 1957.

[19] James Joseph Sylvester (1814-1897), British mathematician; taught in Europe and America; contributed to algebra.

[20] Philip Jourdain, obituary of Poincaré in *The Monist*, vol.22, 1912, pp.611 *et seq.*

[21] Nikolai I. Lobachevskii (1793-1856) and Janos Bolyai (1802-1860), Russian and Hungarian mathematician respectively; made fundamental contributions to non-Euclidean geometry.

[22] Vito Volterra, *Henri Poincaré*, Rice Institute pamphlet, vol.1, no.2, pp.113-162 (1915).

[23] George David Birkhoff (1884-1944), American mathematician; studied at the University of Chicago, and taught at Harvard.

[24] Jean-Pierre Serre, *Cours D'Arithmêtique*, Presses Universitaites de France, Paris, 1970.

[25] Stephen Smale (1930-), American mathematician; solved the generalized Poincare conjecture for manifolds of dimension n > 4, *(Annals of Mathematics*,

vol.74, 1961, pp.391-406); one of the four 1966 recipients of the Fields' Medal (equivalent to the Nobel prize in other sciences, awarded once every four years to young mathematicians who have accomplished significant works); studied at Michigan, now professor at the University of California, Berkeley.

[26] Karl Weierstrass (1815-1897), German mathematician; one of the founders of mathematical analysis; contributed to the theory of functions.

[27] Josiah Royce (1855-1916), American philosopher; studied at Johns Hopkins University and taught at Harvard; a pioneer of the American school of philosophy.

[28] Translated by G. B. Halsted; published by the Science Press, Lancaster, Pennsylvania, 1913.

[29] *Science and Hypothesis*.

[30] *The Value of Science*.

[31] Emmanuel Kant (1724-1804), German philosopher; held that there was but one 'true' geometry — the Euclidean geometry.

[32] Georg Cantor (1845-1918), German mathematician; best known for the invention of set theory; suffered several nervous breakdowns due to opposition of several mathematicians to his theory; died in a mental hospital.

[33] Ernst Zermelo (1971-1953), German mathematician; with A.A. Fraenkel introduced axiomatic set theory — the Zermelo-Fraenkel theory of sets, as is now commonly known; studied at Göttingen under Hilbert.

[34] Abraham A.Fraenkel(1891-1966), taught at the Hebrew University of Jerusalem and set up a strong school of logic there.

[35] Kurt Gödel (1907-), German mathematician; studied at the University of Vienna; made fundamental contributions to mathematical logic; probably the greatest living logician today; emigrated to America during second world war; became a permanent member of the Institute for Advanced Study, Princeton.

[36] John von Neumann (1903-1957), Hungarian-born mathematician; contributed to various branches of pure and applied mathematics and computer science; became a permanent member of the Institute for Advanced STudy, Princeton. See Y. K. Leong, "Notes on mathematicians: 3. John von Neumann", this *Medley*, vol.3, no.3, 1975, 90-106.

[37] Bertrand Russell (1872-1970),with Whitehead considered two of the most influential British philosophers of the present century; their joint work *Principia Mathematica* attempted to formalize mathematics and to derive every mathematical truth from a set of logical axioms; taught at Cambridge and received Nobel prize for literature in 1952.

[38] Alfred North Whitehead (1861-1947), taught Russell at Cambridge, and later became professor at Harvard, where he died.

[39] Giuseppe Peano (1858-1932), Italian mathematician; introduced the Peano axioms for number theory.

[40] For example, the "set" of all sets which are not members of themselves is both a member of itself and not a member of itself.

[41] Charles Hermite (1922-1905), French mathematician; contributed to algebra and number theory.

[42] "Mathematical creation", in *Science and Method*.

[43] Now known as automorphic functions; named Fuchsian by Poincaré in honour of the German analyst, Lazarus Fuchs (1833-1902).

* * * * * * * * * *

For the student it is very instructive to learn not only the final result, the final formulation, but also the history of its development. For in this way he not only gains an insight into the process of intellectual development, but also realizes that the difficulties which he may encounter in assimilating the new ideas are not necessarily due to inferiority on his part, but rather to the high degree of sophistication required for grasping the ideas in question. Realizing the misfortunes of his predecessors, he is less discouraged by his own.

— Antoni Zygmund (1900-)