ANOTHER DEFINITION OF A GROUP

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A group is usually defined as an algebraic system which is a semigroup, i.e. a set of elements together with an associative binary operation, in which certain equations always have solutions. One such condition is:

For any a, b in G there exist x, y in G such that ax = b and ya = b.

Another common formulation is:

There exists e in G such that for every a in G, ae = a = ea; and

For every a in G there is b in G such that ab = e = ba.

There is some theoretical interest in the way in which these conditions may be further simplified. For example, it was shown in [1] that a commutative semigroup is a group if it satisfies

(i) For every a there is an x such that xa = a; such an x is called a local left identity for a, or more briefly, an l.l.i. of a; and

(ii) For every a and l.l.i. x there is a left inverse of a relative to x, i.e. there is b such that ba = x. It should be noted that it is not assumed in (i) that x is the same for every a; while in (ii) b may depend on both a and x. Of course, in the case that both conditions are satisfied, this dependence is only apparent, since then x will be <u>the</u> identity of the system and b will be <u>the</u> inverse of a.

It was noted by P. J. Sally in [2] that the full force of the commutativity condition is not used, and that it may be replaced by

(iii) Every l.l.i. is central; i.e. for any l.l.i. x of some a and for every b, xb = bx.

We show that (iii) may be simplified still further and, in this connexion, a few words of explanation about the sense in which simplification is used may be in order. We envisage that the system is given in such a way that for any a, b we can determine the c such that ab = c; for example, in the case of a finite system, that we are given the multiplication table (and that distinct symbols denote distinct elements). In order to check whether the system is a group we have to compute various products and test for associativity, the existence of an identity, etc. Then the postulates are simplified if the number of such computations is reduced. Local conditions, as against universal ones, clearly have an advantage in this respect. We shall also derive some negative results.

The terms local right identity (l.r.i.), local identity (l.i.), identity, and left or right identity have their standard meaning.

It is clear that (iii) (or commutativity) implies that every l.l.i. is indeed an l.i. It follows from (i) and (iii) that every element has an l.r.i. and we now show that every such is also an l.i. Take any a and let u be an l.r.i. of a. Now let x, a' be such that xa = a and a'a = x. We then have

ua = uax = xua = a'aua = a'aa = xa = a,

so u is an l.l.i. of a. This leads to

THEOREM. A semigroup S is a group if it satisfies (i), (ii) and

(iii') Every local one-sided identity of an element is a local identity of that element.

Proof. We must show that if a, x are such that either xa = a or ax = a, then xa = ax. We follow the ideas of [1] and show first that each element has a unique l.l.i. So let x, y be **l.l.i.'s** of a and let a' be such that a'a = x. We then have

$$xy = a'ay = a'a = x.$$

So y is an l.r.i. of x (and if y is x, that x is an idempotent). By (iii'), xy = yx, and a similar argument now shows that yx = y. Therefore x = y. Thus every element has a unique l.l.i. and since this is also l.r.i. it is clear that each element has a unique local identity.

Now let a, b be arbitrary elements and x, y their respective local identities. Putting t = ab, we get

$$xt = xab = ab = aby = ty,$$

so that x is an l.l.i. and y an l.r.i. of t. By what was proved above it follows that x = y. Hence all the local identities are the same and S has an identity. (I am indebted to Dr. U. C. Guha for putting the argument in this form - an earlier version made use of the fact that a l.l.i. is idempotent.)

Condition (ii), together with the existence of the identity, immediately imply that S is a group.

The proof suggests another way of modifying the conditions, viz. to replace (i) and (iii') by

(i') Every element has a local identity which is also the unique one-sided identity of that element.

However, it is not evident that this is a simplified condition in the earlier sense. It is **clear** that (i') alone is sufficient to ensure the existence of the identity.

We conclude with a pair of examples which show that the theorem fails if the conditions are further weakened in a way to be indicated. We proceed in a concrete way and both examples are subsemigroups of the semigroup of all mappings of the set $A = \{0,1\}$ into itself, with composition of functions as the operation. This automatically ensures associativity and the distinctness of the elements.

We require the three mappings I, z, u which map $(0,1)$ onto $(0,1)$, $(0,0)$ and $(1,1)$, respectively. The following relations are easily verified:

$$I^2 = I, \; Iz = zI = z \; ;$$

$$zu = z^2 = z, \; uz = u^2 = u \; .$$

Thus each of mappings is idempotent, and hence, is a local identity of itself. It is also its own left and right inverse relative to itself as an l.i.

The subsemigroup $S_1 = \{I,z\}$ clearly satisfies (i) and (iii'), but (ii) is not satisfied since z has no left inverse relative to the l.i. I. The following weaker form of (ii) is satisfied, viz.

(ii') Every element has a left inverse relative to at least one of its l.l.i.'s.

For our second example we take $S_2 = \{x,u\}$. Then (i) and (ii) are satisfied but (iii') is not, since each element is an l.r.i. of the other and is not an l.l.i. This shows that (iii') cannot be replaced by the condition that every l.l.i. is an l.i. It is clear that S_2 is not a group. In the same way (i') cannot be replaced by the condition that every element has a unique local identity.

We have not considered the effect of weakening (i) since there does not seem to be any way of doing so that would leave the issue in doubt. Nor have we seen any way of "localising" the condition of associativity.

References

[1] W.E. Deskins and J. D. Hill, On the definition of a group, *Amer. Math. Monthly*, 68 (1961) 795-6.

[2] P. J. Sally, Problem 5066, *Amer. Math. Monthly*, 70 (1963) 96.

A circle is a happy thing to be —
Think how the joyful perpendicular
Erected at the kiss of tangency
Must meet my central point, my avatar.
And lovely as I am, yet only 3
Points are needed to determine me.

— Christopher Morley (1980-)