PROBLEMS AND SOLUTIONS

A book-voucher prize will be awarded to the best solution of a starred problem. Only solutions from Junior Members and received before 1 March 1976 will be considered for the prizes. If equally good solutions are received, the prize or prizes will be awarded to the solution or solutions sent with the earliest postmark. In the case of identical postmarks, thewinning solution will be decided by ballot.

Problems or solutions should be sent to Dr. Y.K. Leong, Department of Mathematics, University of Singapore, Singapore 10. Whenever possible, please submit a problem together with its solution.

*P7/75. If $0<x<\frac{1}{4}$, find the sum of the infinite series

$$1 + \binom{2}{1}x + \binom{4}{2}x^2 + \binom{6}{3}x^3 + \ldots + \binom{2r}{r}x^r + \ldots ,$$

where $\binom{2r}{r}$ is the binomial coefficient $(2r)!/(r!)^2$.

(Chan Sing Chun)

*P8/75. If the equation $x^3 - ax^2 + bx - c = 0$ has 3 real roots, show that

$$64b^3 \leq 27(ab - c)^2.$$

(via Y.K. Leong)

*P9/75. Show that the following system of equations cannot have real solutions in w, x, y, z :

$$\begin{aligned} x^2 + yz &= 0, \\ w^2 + yz &= 0, \\ xy + yw &= 1, \\ xz + zw &= 1. \end{aligned}$$

(H.N. Ng)

*P10/75. Find all non-negative integers a, b, c such that

$$a^3 + 2b^3 + 4c^3 - 6abc = 1.$$

(H.N. Ng)

Miss Hooi Lai Ngoh has been awarded the prizes for submitting correct solutions to P5/75 and P6/75.

Solutions to P4 - P6/75.

*P4/75. Let A and B be two points inside a circle such that they are equidistant from the centre of the circle. Find a point P on the circumference of the circle such that AP + BP is minimum. Is the point P unique? Calculate the minimum value of AP + BP.

(Louis H. Y. Chen)

Solution by Louis H.Y. Chen, C.T. Chong and H.N. Ng. See "A shortest path problem", This *Medley*, Vol.3, No.3, p. 126.

*P5/75. If α and β are real numbers, prove that $2(\sin^2\alpha+\sin^2\beta) \geq \sin^2(\alpha+\beta)$.

(Stephen T.L. Choy)

Solution by Hooi Lai Ngoh.

Now $2(\sin^2\alpha+\sin^2\beta)=(\sin^2\alpha+\cos^2\alpha+\sin^2\beta+\cos^2\beta)(\sin^2\alpha+\sin^2\beta)$

$$=\sin^4\alpha+\sin^2\alpha\cos^2\alpha+\sin^2\alpha\sin^2\beta+\sin^2\alpha\cos^2\beta$$

$$+\sin^2\alpha\sin^2\beta+\sin^2\beta\cos^2\alpha+\sin^4\beta+\sin^2\beta\cos^2\beta.$$

But $\sin^4\alpha+\sin^2\alpha\cos^2\alpha+\sin^4\beta+\sin^2\beta\cos^2\beta = \sin^2\alpha+\sin^2\beta$,

and $\sin^2\alpha+\sin^2\beta \geq 2|\sin\alpha||\sin\beta|$

$$\geq 2|\sin\alpha||\sin\beta||\cos\alpha||\cos\beta|$$

$$\geq 2\sin\alpha\sin\beta\cos\alpha\cos\beta .$$

Hence $2(\sin^2\alpha+\sin^2\beta)=2\sin^2\alpha\sin^2\beta+\sin^2\alpha\cos^2\beta+\sin^2\beta\cos^2\alpha$

$$+\sin^2\alpha+\sin^2\beta$$

$$\geq 2\sin^2\alpha\sin^2\beta+\sin^2\alpha\cos^2\beta+\sin^2\beta\cos^2\alpha$$

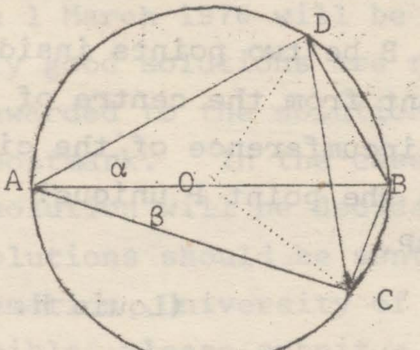
$$+2\sin\alpha\sin\beta\cos\alpha\cos\beta$$

$$=2\sin^2\alpha\sin^2\beta + \sin^2(\alpha+\beta)$$

$$\geq \sin^2(\alpha+\beta).$$

Alternative solution by Chan Sing Chun.

Since $\sin^2\alpha = \sin^2(\pi - \alpha) = \sin^2(\alpha - \pi) = \sin^2(2\pi - \alpha)$, we only need to prove the inequality for $0<\alpha,\ \beta<\frac{1}{2}\pi$.



Draw a circle with centre O and unit diameter. Let AB be a diameter and C, D points on the circumference and on opposite sides of AB such that $D\hat{A}B = \alpha$, $B\hat{A}C = \beta$. Then it is easily seen that

$$DB = \sin\alpha, \quad BC = \sin\beta, \quad CD = \sin(\alpha + \beta).$$

In ΔBCD, $\quad DB + BC \geqslant CD$,

i.e. $\quad \sin\alpha + \sin\beta \geqslant \sin(\alpha + \beta)$.

Since the numbers are non-negative, we get, on squaring,

$$\sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta \geqslant \sin(\alpha + \beta).$$

But $\quad \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta \geqslant 0.$

Adding the inequalities gives the result.

Alternative solution. Consider the complex number

$$(\cos\alpha + i\cos\beta)(\sin\alpha + i\sin\beta)$$

$$= (\sin\alpha\cos\alpha - \sin\beta\cos\beta) + i\sin(\alpha + \beta).$$

Taking the modulus on each side, we have

$$(\cos^2\alpha + \cos^2\beta)(\sin^2\alpha + \sin^2\beta)$$

$$= \tfrac{1}{4}(\sin 2\alpha - \sin 2\beta)^2 + \sin^2(\alpha + \beta).$$

The inequality then follows since $\cos^2\alpha + \cos^2\beta \leqslant 2$ and $\frac{1}{4}(\sin 2\alpha - \sin 2\beta)^2 \geqslant 0$.

*P6/75. If n is a positive integer, prove that

$$\sum_{k=0}^{n} \frac{(-1)^k}{(n+k)!(n-k)!} = \frac{1}{2(n!)^2}.$$

Solution by Hooi Lai Ngoh. Putting x = 1 in the expansion

$$(1-x)^{2n} = \binom{2n}{0} - \binom{2n}{1}x + \binom{2n}{2}x^2 - \ldots + \binom{2n}{2n}x^{2n},$$

we have

$$0 = \binom{2n}{0} - \binom{2n}{1} + \binom{2n}{2} - \ldots + (-1)^{n-1}\binom{2n}{n-1} +$$

$$(-1)^n\binom{2n}{n} + (-1)^{n+1}\binom{2n}{n+1} + \ldots + \binom{2n}{2n-2} - \binom{2n}{2n-1} + \binom{2n}{2n}.$$

Since $\binom{2n}{0} = \binom{2n}{2n}$, $\binom{2n}{1} = \binom{2n}{2n-1}$, etc., we have

$$(-1)^n\binom{2n}{n} = (-1)^n 2\left[\binom{2n}{n} - \binom{2n}{n+1} + \binom{2n}{n+2} - \ldots + (-1)^n\binom{2n}{2n}\right].$$

But the given sum is equal to

$$\frac{1}{(2n)!}\left[\binom{2n}{n} - \binom{2n}{n+1} + \binom{2n}{n+2} - \ldots + (-1)^n\binom{2n}{2n}\right] = \frac{1}{2(n!)^2}.$$

Alternative solution by Chan Sing Chun. We make use of the operators Δ and E (see, for example, Aitken and Milne-Thomson, *Finite differences*, or Freeman, *Mathematics for acturial students*). Write

$$n^{(-r)} = \frac{1}{(n+1)(n+2)\ldots(n+r)}, \quad r \geq 1.$$

Then it is easily checked that

$$n^{(-r)} = \frac{(-1)^{r-1}}{(r-1)!}\Delta^{r-1}n^{(-1)}, \quad r \geq 1,$$

and $E^{n-1}(n^{(-1)}) = (2n-1)^{(-1)} = 1/(2n)$.

The given series may be rewritten as

$$\frac{1}{(n!)^2}\left(1 - \frac{n}{n+1} + \frac{n(n-1)}{(n+1)(n+2)} - \frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)} + \ldots\right).$$

Now

$$1 - n\left(\frac{1}{n+1} - \frac{n-1}{(n+1)(n+2)} + \ldots\right)$$

$$= 1 - n(n^{(-1)} - (n-1)n^{(-2)} + (n-1)(n-2)n^{(-3)} - \ldots)$$

$$= 1 - n(n^{(-1)} - (n-1)n^{(-1)} + \frac{(n-1)(n-2)}{2!}{}^2n^{(-1)} - \ldots)$$

$$= 1 - n((1+\Delta)^{n-1}n^{(-1)})$$

$$= 1 - nE^{n-1}(n^{(-1)})$$

$$= 1 - n.1/(2n) = \tfrac{1}{2}.$$

Alternative solution. Consider the identity

$$1 = e^{-x}e^{x} = \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right).$$

Write the series on the right-hand side as a series in x, and look at the coefficient of x^{2n} $(n>1)$.

Then $0 = \frac{1}{(2n)!} - \frac{1}{1!\,(2n-1)!} + \frac{1}{2!\,(2n-2)!} - \cdots +$

$$(-1)^{n-1}\frac{1}{(n-1)!(n+1)!} + (-1)^n\frac{1}{n!n!} +$$

$$(-1)^{n-1}\frac{1}{(n+1)!(n-1)!} + \cdots +$$

$$\frac{1}{(2n-2)!2!} - \frac{1}{(2n-1)!1!} + \frac{1}{(2n)!}.$$

Regrouping the terms gives

$$\frac{1}{(2n)!} - \frac{1}{1!(2n-1)!} + \frac{1}{2!(2n-2)!} + \cdots + (-1)^{n-1}\frac{1}{(n-1)!(n+1)!} +$$

$$(-1)^n\frac{1}{2(n!)^2} = 0.$$

Multiplying by $(-1)^n$ and adding $1/(n!)^2$ yields the required sum of the given series.

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A LETTER TO THE EDITOR

I have a few suggestions to make:

(a) a mathematics workshop be opened whereby members can hold discussions, instructive sessions, problem evaluations, etc., at least once a week. Otherwise disinterest and mental asthenia in the mathematical field may develop due to lack of exercise in active problem-solving and exposure to interesting aspects of mathematics known as "mathematical recreations", which is a very wide field that encompasses all branches of knowledge. A lecture once a month is too static with very little member participation. A lecture room may be assigned exclusively for the use of the Society at approved hours of the day or night and for members to congregate and discuss subjects of common interest. In this connection, adequate means of communication between members is vital. The telephone and the surface mail are two of the best ways of doing it. There should be more intimate relations between members as far as mathematical discussions are concerned, because some day mathematical discoveries may result from these discussions.