MAGIC SQUARES AND DOODLING*

Lee Peng-Yee
Nanyang University

Mathematical research is not something mysterious. Everyone can do it. To prove my point, I shall give two examples. The first one concerns magic squares, and the second one doodling.

Our problem is how to construct a magic square of size 100 by 100. For those who are interested in the history of the subject, there is a short but nonetheless comprehensive account of it in [1] . Recently, this has become an active research topic. For some new results, see, for example,[2]. Here we try to show how we may proceed from special to general and hence devise a way of constructing the magic square.

Let us begin with 3 by 3 magic squares. It is well-known that there is only one, namely the following

2	9	4
7	5	3
6	1	8

For 4 by 4, we simply write down 1 to 16 in the order shown on the left hand side below. Then we reflect the diagonals and the result is a 4 by 4 magic square as shown on the right below.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

*This is the text of a public lecture delivered on 28 August 1975.

Note that any row, any column or any of the two diagonals all add up to 34. We sometimes call 34 the magic number for the magic square. It is very easy to find the magic number for a magic square. We simply fill up the square as we did on the left above; then add up the diagonal. For example, the magic number for a 5 by 5 magic square is 1 + 7 + 13 + 19 + 25 = 65.

(1)↘ (2)↓ (3)↙

	1	2	3	4	5
	6	7	8	9	10
(4) →	11	12	13	14	15
	16	17	18	19	20
	21	22	23	24	25

Further note that all the row, column and diagonals having 13 add up to the same sum 65. When we rearrange the numbers we shall keep them together. Take out the arrays of numbers (1), (2), (3) and (4) and replace them as follows.

15	2	21	4	23	
6	14	17	18	10	
25	19	13	7	1	← (1)
16	8	9	12	20	
3	22	5	24	11	

(2)↗ (3)↑ (4)↖

Hence we have obtained a 5 by 5 magic square.

We may carry on, and it seems to get harder each time. Write down a 6 by 6 square and reverse its two diagonals. We have the following:

36	2	3	4	5	31	+30
7	29	9	10	26	12	+18
13	14	22	21	17	18	+ 6
19	20	16	15	23	24	- 6
25	11	27	28	8	30	-18
6	32	33	34	35	1	-30
+5	+3	+1	-1	-3	-5	

differences from the magic number 111

We note that the magic number is 111. Let us find the sum of each row or column and its difference from the magic number. We see that all we need to do is to interchange a number in the first row with a number in the last row (except those on the diagonals), and similarly for the second row and fifth row, third row and fourth row. We should also do the same for the columns. This can be done by interchanging the following:

5 — 35   7 — 12
9 — 27   20 — 23
13 — 19   33 — 34

The result is a 6 by 6 magic square.

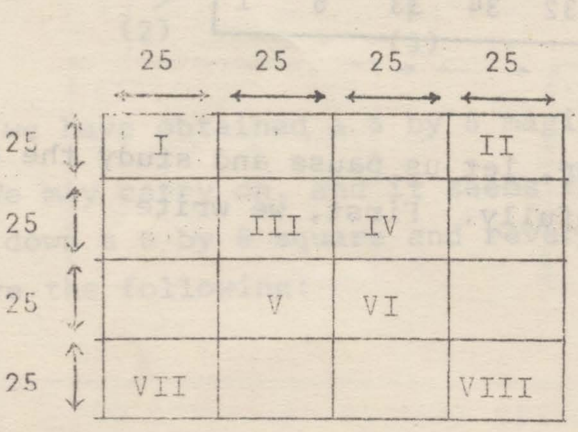
36	2	3	4	35	31
12	29	27	10	26	7
19	14	22	21	17	18
13	23	16	15	20	24
25	11	9	28	8	30
6	32	34	33	5	1

Before we go further, let us pause and study the 6 by 6 magic square more carefully. First, we write

1	2	3	4	5	6	+90
7	8	9	10	11	12	+54
13	14	15	16	17	18	+18
19	20	21	22	23	24	-18
25	26	27	28	29	30	-54
31	32	33	34	35	36	-90
+15	+9	+3	-3	-9	-15	

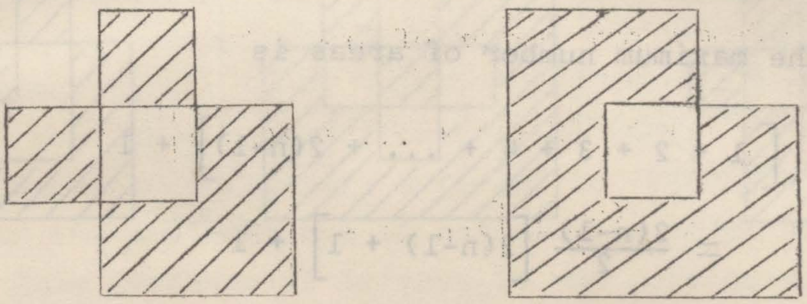
Suppose we work out the sum for each row and column and find its difference from the magic number. Obviously, all we need to do is to interchange three numbers between corresponding rows and columns. In fact, this was what we did before. Note that we interchanged 8 with 29, and 11 with 26. The effect is equivalent to interchanging two numbers between two corresponding rows and two corresponding columns.

Now we have a procedure of constructing any even magic square. First we write down the square as we did before. Work out the magic number and the difference from it for the sum of each row and each column. Decide on the number of entries we need to interchange and finally devise a scheme to interchange them. The reader may wish to try this himself for a 10 by 10 magic square. To construct a 100 by 100 magic square, we proceed as before and find that we need to interchange 50 numbers between the corresponding rows and columns. This may be done by dividing the square into the following blocks:



Then interchange diagonally block I with block VIII, II with VII, III with VI, and IV with V. For example, we interchange 1 with 10000, 2 with 9999 and etc. In fact, this scheme works for any 4n by 4n magic squares. Hence we have solved the first problem in a surprisingly easy way.

We all doodle. The question is what mathematics, if any, we can get out of doodling. Let us assume that we draw only vertical and horizontal lines. We always make full turns and come back to the original point. For example, the following are two different designs of making two turns.



If we keep doodling, we find that the above are the only two designs we can have for two turns. The first one has four enclosed areas (3 blacks and 1 white) and the second one two only (1 black and 1 white).

If we make three turns, either we keep crossing whenever we can or we try to avoid it whenever we can. The former gives 1 + 2 + 3 + 4 = 10 crossings, whereas the latter only 2. Whenever we cross once, we obtain an enclosed area. When we complete the doodling we add an extra area. So we have the following so-called doodling theorem:

The number of enclosed areas is equal to the number of crossings plus one.

Therefore, if we make three turns, the maximum number of enclosed areas we can have is 11 and the minimum is 3.

We can generalize this to any number of turns. The minimum case is easy. Let the number of turns be n. Then the minimum number of areas is also n. For the maximum case, let us study the following table:

turns	maximum crossing	maximum areas
1	0	1
2	1 + 2	4
3	1+2+3+4	11
4	1+2+3+4+5+6	22
5	1+2+...+7+8	37

Therefore the maximum number of areas is

$$\left[1 + 2 + 3 + 4 + \ldots + 2(n-1)\right] + 1$$

$$= \frac{2(n-1)}{2}\left[2(n-1) + 1\right] + 1$$

$$= 3n^2 - 3n + 2$$

Another question we may ask is how many different designs there are for three turns. I have found 16 of them (see Appendix). A research problem is the following. Can you find another one which is different from the sixteen or can you show that you cannot?

o

References

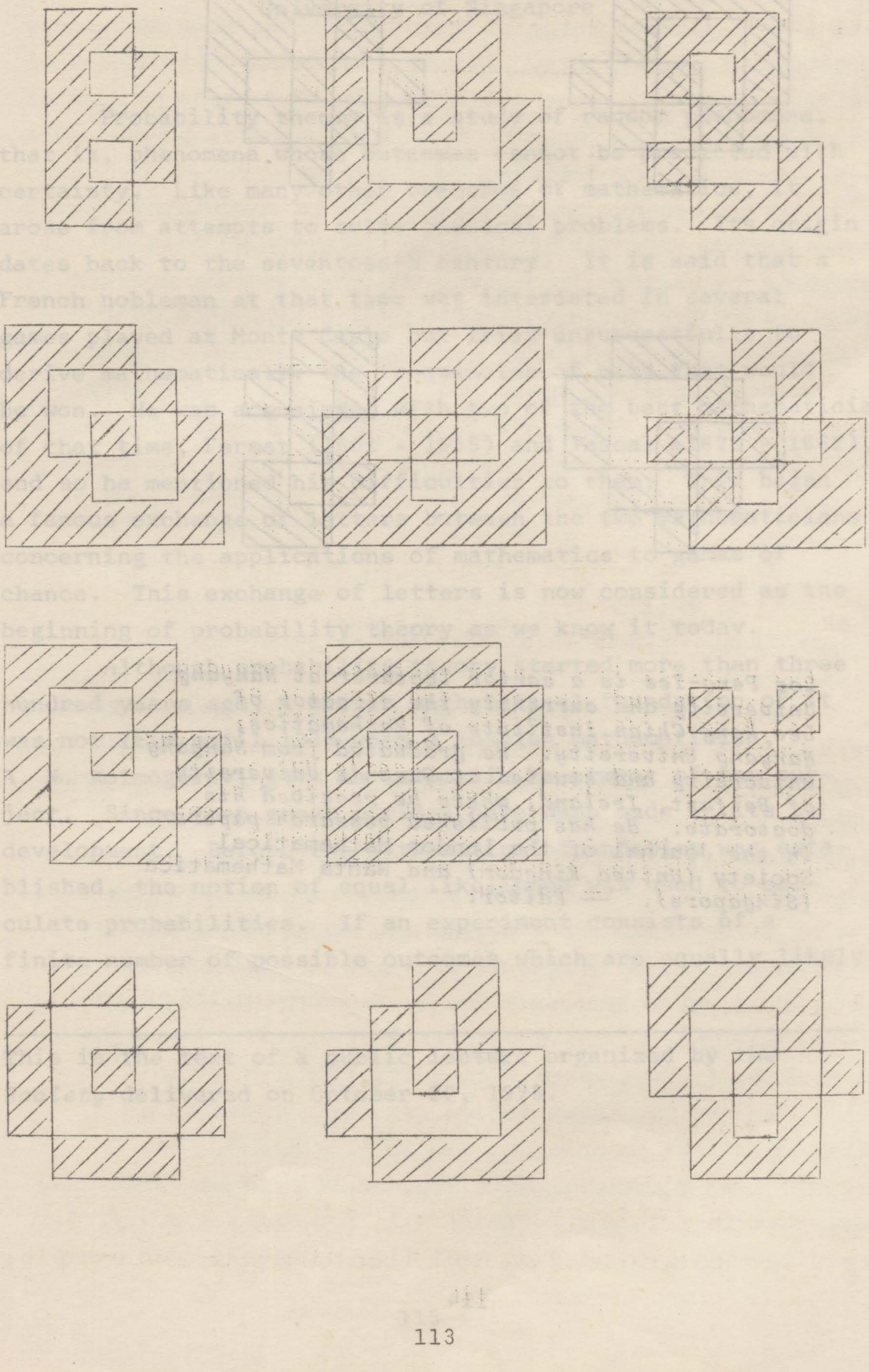
[1] *Encyclopedia Britannica*, 15th Edition (William Benton, 1974)

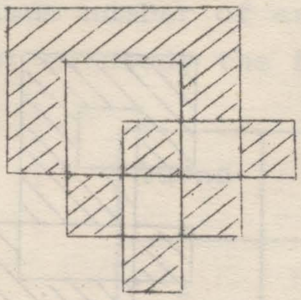
[2] J. Dénes and A.D. Keedwell, *Latin squares and their applications*, (English Universities Press, 1974)

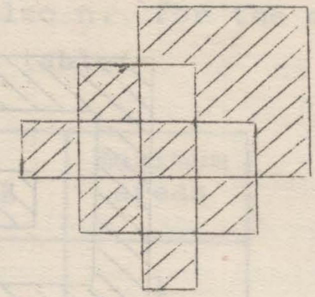
[3] Lee Peng-Yee, "Construction of 100 by 100 magic squares", to appear in *Bull. Malaysian Math. Soc.*

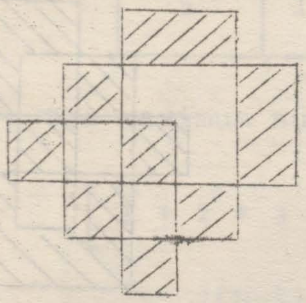
[4] ___, "Mathematical theory of doodling", to appear in *Bull. Malaysian Math. Soc.*

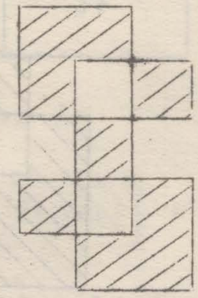
APPENDIX











Lee Peng-Yee is a senior lecturer at Nanyang University and currently the director of Lee Kong Chian Institute of Mathematics, Nanyang University. He graduated from Nanyang University and studied at Queen's University of Belfast, Ireland, where he received his doctorate. He has published research papers in the Journal of the London Mathematical Society *(United Kingdom) and* Nanta Mathematica *(Singapore). — Editor.*