A SHORTEST PATH PROBLEM

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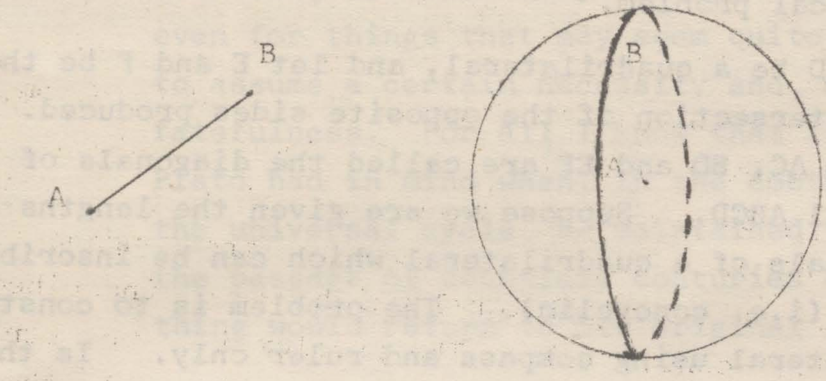
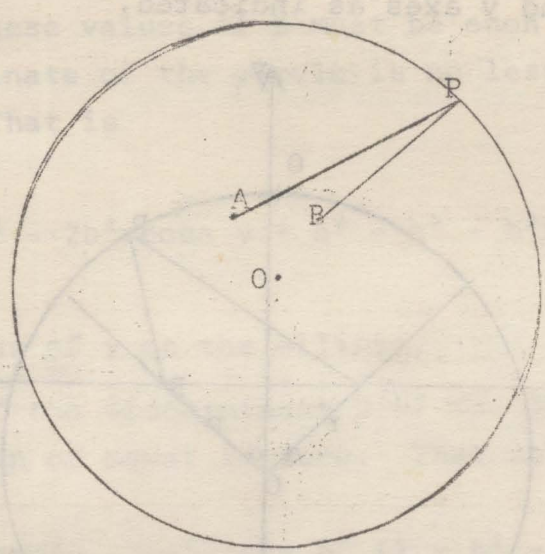
Problems on shortest paths are interesting and very easy to state. Of course in euclidean geometry one always has the straightline as the shortest distance joining two given points. On a sphere, the shortest path to travel from one point to another would be the shorter of the two arcs which lie on the great circle containing the two points, and which have these two points as their endpoints.



A shortest path problem which evolved from Question 2, Paper 2 of the 1975 Interschool Mathematical Competition [1] and which later appeared under P4/75 [2] is the following:

> *Let* P *be a point on the circumference of a circle and let* A *and* B *be two points inside the circle such that they are equidistant from the centre. Find a position of* P *such that* AP+BP *is minimum.*

One can further ask whether this position of P is unique and that the minimum value of AP+BP be computed.

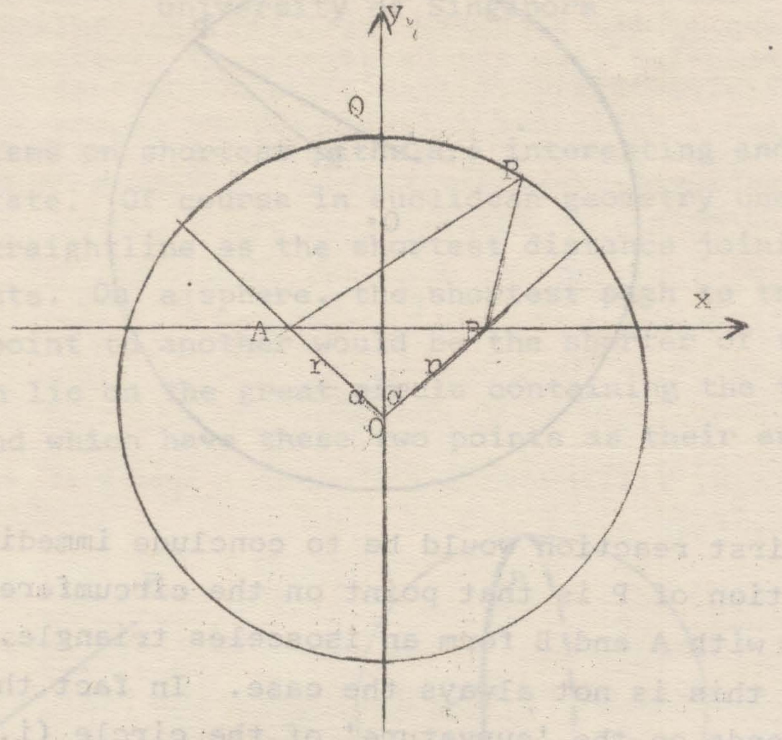


A first reaction would be to conclude immediately that the position of P is that point on the circumference which together with A and B form an isosceles triangle. It turns out that this is not always the case. In fact the position of P depends on the 'curvature' of the circle (i.e. how **it** 'curves') and therefore the radius of the circle, and also on the angle intercepted by AOB. A solution is provided in this note.

It is easy to see that if E_o is an ellipse, with A and B as foci, which lies inside and touches the given circle at P_o, then P_o is a position of P for whcih AP+BP is minimum. Furthermore, AP_o+BP_o is equal to the major axis of E_o. Thus the problem reduces to finding such an ellipse (if it exists) and locating P_o. We shall do the problem by analytic geometry.

For simplicity, we assume the circle to have unit radius. Let the distance of A and B from the centre O be r and let $\angle$ ABC = 2α . Without loss of generality, we may assume that $0 < \alpha \leq \frac{\pi}{2}$.

Choose the x and y axes as indicated.



Then A ≡ (-rsinα,0) and B ≡ (rsinα,0).

The equation of the given circle is

$$(1) \qquad x^2 + (y + r\cos\alpha)^2 = 1$$

and that of an ellipse with A and B as foci and b as the semi-minor axis is

$$(2) \qquad \frac{x^2}{b^2+r^2\sin^2\alpha} + \frac{y^2}{b^2} = 1 .$$

Note that $a = \sqrt{b^2+r^2\sin^2\alpha}$ is the semi-major axis.

Since we shall be interested in only those ellipses which lie within the circle (1), we shall first determine those values of b for which the ellipse (2) lies in the circle (1). These values of b must be such that the square of the x-coordinate of the circle is no less than that of; the ellipse. That is

(4) $r^2\sin^2\alpha\, y^2 - 2b^2 r\cos\alpha\, y + b^2 - b^4 - b^2r^2 \geqslant 0$

for those values of y on the ellipse.

This is true if the discriminant D of the left hand side of (4) is less than or equal to zero. That is

$$D = 4b^4r^2\cos^2\alpha - 4r^2\sin^2\alpha\, b^2\,(1 - b^2 - r^2)$$

$$= 4b^2r^2\left[\, b^2 - (1 - r^2)\sin^2\alpha \,\right] \leqslant 0$$

Hence the ellipse (1) lies in the circle (2) if

(5) $$b^2 \leqslant (1 - r^2)\sin^2\alpha \, .$$

We need to consider two cases.

Case (1) $r > \cos\alpha$.

For this case, we take $b^2 = (1 - r^2)\sin^2\alpha$.

Then (2) becomes

(6) $$\frac{x^2}{\sin^2\alpha} + \frac{y^2}{(1-r^2)\sin^2\alpha} = 1 \, .$$

We shall show that (6) is the equation of E_0. By (5), the ellipse (6) lies in the circle (1). Solving (1) and (6), we obtain the following real roots

(7) $$\begin{cases} x = \pm\sqrt{1 - \dfrac{\cos^2\alpha}{r^2}} \\ y = \dfrac{(1-r^2)\cos\alpha}{r} \end{cases}$$

Hence (6) is the equation of E_o which touches the circle at two distinct points whose coordinates are given by (7) and $AP_o + BP_o = 2a = 2\sin\alpha$.

Case (2) $r \leqslant \cos\alpha$ (This is possible only if $0 < \alpha < \frac{\pi}{2}$).

Consider the ellipse with foci A and B which passes through Q. Then it has the equation

$$(8) \qquad \frac{x^2}{1-2r\cos\alpha + r^2} + \frac{y^2}{(1-r\cos\alpha)^2} = 1$$

Since
$$\begin{aligned} b^2 &= (1-r\cos\alpha)^2 \\ &= (r-\cos\alpha)^2 + (1-r^2)\sin^2\alpha \\ &\geqslant (1-r^2)\sin^2\alpha , \end{aligned}$$

we cannot make use of (5).

However, the left hand side of (4) has two real roots, namely,

$$(9) \qquad y = \frac{1-r\cos}{r\sin^2\alpha}\left[\cos\alpha - r\cos^2\alpha \pm (\cos\alpha - r)\right]$$

of which the smaller root is $1-r\cos\alpha$ which corresponds to Q. Therefore, (4) is satisfied for those values of y on the ellipse and this shows that the ellipse (8) lies in the circle (1). Furthermore, (9) implies that (1) and (8) have only one root, namely,

$$(10) \qquad \begin{cases} x = 0 \\ y = 1 - r\cos\alpha . \end{cases}$$

Hence, (8) is the equation of E_o which touches the given circle (1) at only one point (namely Q) whose coordinates are given by (10) and $AP_o + BP_o = AQ + BQ$

$$= 2a = 2\sqrt{1-2r\cos\alpha+r^2}.$$

References

[1] 1975 inter-School Mathematical Competition, This *Medley*, vol.3, No.2 (1975), 61-77.

[2] Problems and Solutions, ibid., 79-82.

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Age is of course a fever chill
That every physicist must fear.
He's better dead than living still,
When once he's past his thirtieth year!

— attributed to P.A.M. Dirac

THE ONLY SOLUTION

We shall have to evolve
problem-solvers galore -
since each problem they solve
creates ten problems more.

— Piet Hein