Problems and Solutions

A book-voucher prize will be awarded to the best solution of a starred problem. Only solutions from Junior Members and received before 31 October 1975 will be considered for the prizes. More than one solution may be submitted. If equally good solutions are received, the prize or prizes will be awarded to the solution or solutions sent with the earliest postmark. In the case of identical postmarks, the winning solution will be decided by ballot.

Problems or solutions should be sent to Dr. Y. K. Leong, Department of Mathematics, University of Singapore, Singapore 10. Whenever possible, please submit a problem together with its solution.

*P4/75. (Cf. Question 2, Paper 2, Inter-School Mathematical Competition 1975) Let A and B be two points inside a circle such that they are equidistant from the centre of the circle. Find a point P on the circumference of the circle such that AP + BP is minimum. Is the point P unique ? Calculate the minimum valu e of AP + BP.

(Louis H. Y. Chen)

*P5/75. If α and β are real numbers, prove that

$$2(\sin^2\alpha + \sin^2\beta) \geqslant \sin^2(\alpha + \beta).$$

(Stephen T. L. Choy)

*P6/75. If n is a positive integer, prove that
$$\sum_{k=0}^{n} \frac{(-1)^k}{(n+k)!(n-k)!} = \frac{1}{2(n!)^2} .$$

(via Y. K. Leong)

Solutions to P1 - P3/75

P1/75. If n and p are positive integers, show that $(np)!/(n!(p!)^n)$ is an integer.

(Chan Sing Chun)

Solution by Proposer. The number of ways of distributing np unlike objects into n groups, which are linearly ordered and which contain p objects each, is equal to
$$\binom{np}{p}.\binom{np-p}{p}.\binom{np-2p}{p}\ldots\binom{p}{p} = \frac{(np)!}{(p!)^n} .$$

If no regard is paid to the order of the n groups, the number of different distributions is
$$(np)!/((p!)^n n!),$$
which must be an integer.

*P2/75. If A and B are any two n×n matrices, prove that $AB - BA \neq I_n$, where I_n is the n×n identity matrix.

(P. H. Diananda)

Solution. Let $A = (a_{ij})$, $B = (b_{ij})$. Consider the trace of a matrix, i.e. the sum of the diagonal entries. Since $AB = (\sum_{k=1}^{n} a_{ik}b_{kj})$, the trace of AB is $\sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk}b_{kj}$, which is also the trace of BA. Hence the trace of $AB - BA$ is zero. But the trace of I_n is $n > 0$. Therefore $AB - BA \neq I_n$.

*P3/75. Let a, b, c, d, e be any real numbers and $d \neq 0$. Prove that the equation

$$x^3 + (a + b + c)x^2 + (ab + bc + ca - d^2)x + e = 0$$

has at least two distinct roots.

(H. N. Ng)

Solution by Proposer. Assume that the equation does not have distinct roots. Since a polynomial of degree 3 has at least one real root, we must then have

$$x^3 + (a + b + c)x^2 + (ab + bc + ca - d^2) + e = (x - u)^3$$

for some real number u. Comparing coefficients, we have

$$-3u = a + b + c$$

$$3u^2 = ab + bc + ca - d^2.$$

Hence $(a + b + c)^2 = 3(ab + bc + ca - d^2)$, i.e. $a^2 + b^2 + c^2 - ab - bc - ca + 3d^2 = 0$, so that

$$\tfrac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2] + 3d^2 = 0.$$

This is impossible since $d \neq 0$. Hence the equation must have distinct roots.

P6/74. If x, y, k are positive numbers with $x \neq y$, $k > 1$, prove that

(i) $$\frac{x}{x + ky} + \frac{y}{y + kx} > \frac{2}{k + 1}$$

(ii) $$\frac{x}{y + kx} + \frac{y}{x + ky} < \frac{2}{k + 1}$$

(Leonard Y. H. Yap)

Solution by Chan Sing Chun. (i) We have $k(k-1)(x-y)^2 > 0$, i.e. $k(k-1)(x^2+y^2-2xy) > 0$. We may rewrite this as

$$(k+1)(kx^2+2xy+ky^2) > 2(kx^2+ky^2+xy+k^2xy),$$

or $(k+1)(x(y+kx)+y(x+ky)) > 2(x+ky)(y+kx)$.

Dividing by $(k+1)(x+ky)(y+kx)$ gives the desired inequality.

(ii) We have $(k-1)(x-y)^2 > 0$, i.e. $(k-1)(x^2+y^2-2xy) > 0$. Rewrite this as

$$2(kx^2+ky^2+(k^2+1)xy) > (k+1)(x^2+y^2+2kxy),$$

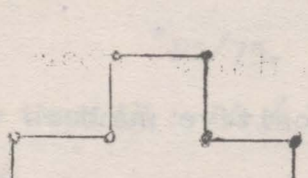
or $2(x+ky)(y+kx) > (k+1)(x(x+ky)+y(y+kx))$.

Dividing by $(k+1)(x+ky)(y+kx)$ gives the desired result.

P10/74. Twelve matches, each of unit length, form the figure of a cross as shown. Rearrange the matches in such a way as to cover an area of four square units.

A solution has been given in this Medley, Vol.3, No.1, April 1975, p.13. An interesting alternative solution has been received from Tay Yong Chiang of Raffles Institution.

Solution by Tay Yong Chiang



(From 'Figures for Fun' by Ya. Perelman)