[3] C. T. Chong, Notes on Mathematicians: 1. Carl Friedlich Gauss, This Medley, vol. 3, No. 1 (1975), 6 - 10.

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[5] C. Reid, Hilbert, Springer Verlag, Berlin, 1970.

[6] H. Vandiver, Fermat's Last Theorem: Its History and the Results Concerning It, Amer. Math. Monthly, vol. 53 (1946), 555 - 578; Supplement in the same journal, vol 60 (1953), 164 - 167.

1975 Inter-school Mathematical Competition+

The 1975 Inter-school Mathematical Competition problems are reproduced below. In Paper 1, an asterisk is marked after each correct answer. In Paper 2, solutions of some of the problems or parts thereof are given.

Paper 1

Monday, June 2, 1975 Time 9.00 a.m. - 11.00 a.m.

1. The number $\sqrt{2}$ is

   (a) a rational fraction; (b) a finite decimal; (c) 1.41421;

   (d) an infinite repeating decimal; (e)* an infinite non-repeating decimal.

2. If n is a positive integer greater than or equal to 2,

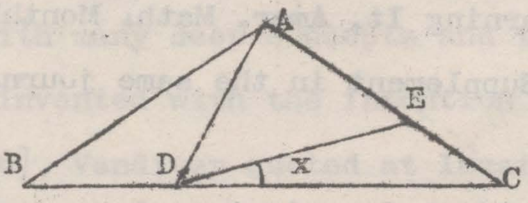
+The Society wishes to thank all those who in one way or another helped to make this competition a success, paticularly the members of its 1975 Competition Subcommittee, comprising Dr. Louis H. Y. Chen (University of Singapore), Mr. Chong Tien Hoo (Institute of Education), Dr. Lee Peng Yee (Nanyang University), Dr. Ng How Ngee (University of Singapore), Dr. Shee Sze Chin (Nanyang University) and Mr. Yeo Kok Keong (Ministry of Education).

then $n^2(n^2 - 1)$ is always divisible by

(a)* 12; (b) 24; (c) any multiple of 12; (d) 12 − n; (e) none of the above.

3. In the figure, $\overline{AB} = \overline{AC}$, angle BAD = 30° and $\overline{AE} = \overline{AD}$. Then x equals

(a) $7\frac{1}{2}°$; (b) 10°; (c) $12\frac{1}{2}°$; (d)* 15°; (e) 20°.



4. The hypotenuse of a right angled triangle is 9cm. and the radius of the inscribed circle is 1 cm. The perimeter of the triangle in cm is

(a) 12; (b) 19; (c)* 20; (d) 24; (e) none of the above.

5. In a circle with centre O, the chord AB is produced to C so that $\overline{BC}$ equals the radius of the circle. CO is drawn and extended to meet the circumference at D. AO is drawn. Which of the following statements is correct where $\angle AOD = x$ and $\angle ACO = y$?

(a)* $x = 3y$; (b) $x = 2y$; (c) $x = 60°$; (d) $x = 2y$ or $x = 3y$ depending on the length of AB; (e) there is no special relationship between x and y.

6. The centres of two circles are 41 cm apart. The radius of the smaller circle is 4 cm and that of the larger circle is 5 cm. The length of the common internal tangent is

(a) 41 cm; (b) 39 cm; (c) 39.8 cm; (e)* 40 cm.

7. The function $x^2 + px + q$ with p and q greater than 0 has its minimum value when

(a) $x = -p$; (b) $x = -2p$; (c) $x = p^2/4q$; (d) $x = p/2$; (e)* $x = -p/2$.

8. The arithmetic mean (that is, the average) of the first n positive integers is

(a) $n/2$; (b) $n^2/2$; (c) $(n - 1)/2$; (d)* $(n + 1)/2$; (e) n.

9. On the same set of axes are drawn the graphs of $y = ax^2 + bx + c$ and the graph of the equation obtained by replacing x by $-x$ in the given equation. If $b \neq 0$ and $c \neq 0$, these graphs intersect

(a) in two points, one on the x-axis, and one on the y-axis;

(b) in one point located on neither axis;

(c) only at the origin; (d) in one point on the x-axis;

(e)* in one point on the y-axis.

10. If the equation $ax^2 + 2bx + c = 0$ has two equal roots, then another true statement about a, b, c is:

(a)* they form a geometric progression; (b) they form an arithmetic progression; (c) they are unequal; (d) they are all negative numbers; (e) only b is negative and the other two positive.

11. Which of the following statements is true ?

(a) The set of all rational numbers forms a group under multiplication. (b) The set of all subsets of a set forms a group under union ($\cup$). (c)* The set of all positive integers does not form a group under multiplication. (d) The set $\{ 0,1 \}$ forms a group under multiplication. (e) The set of all integers form a group under multiplication.

12. Which of the following statements is **false** ?

(a) 6 is not a prime number. (b) The equation $2 + x = 2$ has no solution in N. (c)* For all $x, y, z \in \mathbb{Z}$, if $x \cdot z = y \cdot z$, then $x = y$. (d) For all $m, n, p \in N$, if $m + p = n + p$, then $m = n$. (e) Zero (0) is an integer.

13. The sum of the determinants $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$, $\begin{vmatrix} a & e \\ b & f \end{vmatrix}$, $\begin{vmatrix} a & g \\ b & h \end{vmatrix}$, $\begin{vmatrix} a & i \\ b & j \end{vmatrix}$

is

(a)* $\begin{vmatrix} a & c+e+g+i \\ b & d+f+h+j \end{vmatrix}$; (b) $\begin{vmatrix} 4a & c+e+g+i \\ 4b & d+f+h+j \end{vmatrix}$; (c) $\begin{vmatrix} 2a+2b & c+e+g+j \\ 2a+2b & d+f+h+j \end{vmatrix}$;

(d) $\begin{vmatrix} c+e & g+i \\ d+f & h+j \end{vmatrix}$; (e) none of the above.

14. The position vector of A with respect to O is (2,5). If $\overline{AB} = 2\overline{OA}$, then the position vector of B is

(a) (2,5); (b) (2,15); (c) (4,10); (d)* (6,15); (e) (8,20).

15. The position vector of a moving point P at time t is $\vec{s}(t) = 3t\vec{i} + (t^2 - 1)\vec{j} + t^3\vec{k}$, where $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors along the rectangular axes Ox, Oy, Oz respectively. The velocity vector at time t is

(a)* $3\vec{i} + 2t\vec{j} + 3t^2\vec{k}$; (b) $3 + 2t + 3t^2$; (c) $3\frac{d\vec{i}}{dt} + 2t\frac{d\vec{j}}{dt} + 3t^2\frac{d\vec{k}}{dt}$;

(d) $3t\frac{d\vec{i}}{dt} + (t^2 - 1)\frac{d\vec{j}}{dt} + t^3\frac{d\vec{k}}{dt}$; (e) none of the above.

16. Let $s_1 = \sum_{r=1}^{n} r$, $s_2 = \sum_{r=1}^{n} r^2$, and $s_3 = \sum_{r=1}^{n} r^3$. Then the sum of the first n terms of the sequence $\{u_r\}$, where $u_r = (r^2 + 2)(r + 3)$ is

(a) $s_3 + 3s_2 + 2s_1 + 3$; (b) $s_3 + 3s_2 + 2s_1 + 3n$;

(c) $s_1 + s_2 + s_3$; (d) s_1; (e)* none of the above.

17. Let f_1, f_2, f_3, f_4 be functions such that $f_1(x) = 1 - x$, $f_2(x) = \frac{x}{1 - x}$, $f_3 = \frac{x}{x - 1}$, and $f_4(x) = \frac{x - 1}{x}$. Then the composite $f_3 \circ f_4$ equals

(a) f_2; (b) f_3^{-1}; (c) f_4; (d)* f_1; (e) none of the above.

18. The sets A and B have 7 and 8 elements repectively. If their union has 10 elements, how many elements do A and B have in common ?

(a) 3; (b) 2; (c) 1; (d)* 5; (e) none of the above.

19. A population consisting of 2 types of individuals, A and B, is divided into 5 strata. An individual is selected from the population by first selecting at random a stratum

and then selecting an individual at random from the stratum. If Type A and Type B are equal in number, what is the probability that an individual belonging to Type A will be selected ?

(a) 1/5; (b) 2/5; (c) 1/10; (d) 3/10; (e)* none of the above.

20. Which of the following statements is incorrect ?

(a) If A is a subset of B, then the complement of B is a subset of the complement of A. (b) If x and y are both negative, then xy is positive. (c) A function from a set A into a set B is a rule which associates each element of A with a unique element of B. (d) A straight line and a circle can have 0, 1 or 2 points in common. (e)* If $x \geqslant 0$, then $x^2 \geqslant x$.

21. If $a \leqslant b$ and $a' \leqslant b$, then

(a) $aa' \leqslant b^2$; (b) $a + a' \leqslant 2b$; (c) $a - a' \leqslant 0$; (d) $a = a'$; (e)* no ne of the above.

22. Let $A = \{\{a\}, b, \{b\}\}$. Which of the following statements is correct ?

(a) a is a member of A. (b) The intersection of $\{a\}$ and A is $\{a\}$. (c) $\{a,b\}$ is a subset of A. (d)* A has 7 non-empty subsets. (e) A is a 2-element set.

23. Find the maximum value of $x^{1/x}$ where $x > 0$.

(a)* $e^{1/e}$; (b) $(1/e)^e$; (c) e^e; (d) e; (e) none of the above.

24. A sequence is given by $u_n = u_{n-1} + 8$ and $u_{100} = 10$. Then u_n is

(a) 100; (b) 8; (c)* $8n - 790$; (d) u_{n-1}; (e) $n - 90$.

25. Find $\int_1^2 x\sqrt{x-1}\ dx$.

(a) 2/5; (b) 2/3; (c)* 16/15; (d) 1; (e) none of the above.

26. How many functions from $\{a, b, c\}$ into itself are there ?

(a) 9; (b)* 27; (c) 6; (d) 8; (e) none of the above.

27. For what number base is $3 \cdot 4 = 22$?

(a)* 5; (b) 6; (c) 7; (d) 8; (e) 9.

28. Let $f(x) = \sin(x^2)$ and $f'(x)$ its derivative. Find $f'(f'(x))$.

(a) $2x\cos(x^2)$; (b) $4x\cos\{2x\cos(x^2)\}$;

(c) $8x^2\{\cos(x^2)\}\cos^2(x^2)$; (d) $\{\cos(x^2)\}\cos\{2x\cos(x^2)\}$;

(e)* none of the above.

29. Find the limit of the sequence

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}, \ldots$$

(a) $\sqrt{2}$; (b) 1; (c) 3; (d)* 2; (e) none of the above.

30. Evaluate $\displaystyle\int_1^3 \frac{x-2}{1+(x-2)^4}\,dx$.

(a) 4/17; (b) 4/3; (c) 1; (d) 3; (e)* none of these.

31. Sum the series $\dfrac{1}{1\cdot4\cdot7} + \dfrac{1}{4\cdot7\cdot10} + \dfrac{1}{7\cdot10\cdot13} + \cdots$

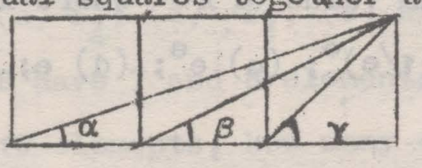
(a) 1; (b) 1/14; (c) 1/2; (d)* 1/24; (e) none of these.

32. Find $\tan(2\pi/5)$.

(a) $\sqrt{5-\sqrt{20}}$; (b)* $\sqrt{5+\sqrt{20}}$; (c) $\sqrt{5}+4$; (d) $\sqrt{5+\sqrt{5}}$;

(e) none of these.

33. Place three equal squares together as shown.



Then

(a) $2\beta = \alpha + \gamma$; (b) $\gamma = 2\beta$; (c) $2\beta = 3\alpha$; (d)* $\alpha + \beta = \gamma$;

(e) $\gamma = 3\alpha$.

34. How many pairs of positive integers a,b are there such that $a^2 - b^2 = 7^2$?

(a) 0; (b)* 1 ; (c) infinitely many; (d) 5; (e) none of these.

35. Let $a_1 > 0$ and $a_n = \frac{1}{2}(a_{n-1} + \frac{1}{a_{n-1}})$. What number does a_n approach ?

(a) a_1; (b) 0; (c)* 1; (d) a_n does not approach any number; (e) none of these.

36. Evaluate $\int_0^{\pi/2} (\frac{\pi}{4} - x)^{20} \sin^2 x \, dx$.

(a) $\sum_{k=1}^{21} \frac{1}{k} (\frac{\pi}{4})^k$; (b) 0; (c) $\sum_{k=1}^{21} \frac{1}{k} \{(\frac{\pi}{4})^k + 1\}$; (d)* $\frac{1}{21} (\frac{\pi}{4})^{21}$;

(e) none of these.

37. Let n be a positive integer. Then the fraction $(n + 9)^2/(n + 3)$ is a positive integer

(a) if n is a multiple of 3; (b) if n is a multiple of 3 not more than 33; (c) if the values of n form an arithmetic progression; (d) for exactly five (5) values of n;

(e)* none of these.

38. A farmer has erected an L-shape fence of dimensions 60 metres by 20 metres. If he has an additional 120 metres of fencing, what is the largest possible rectangular area he can enclose ?

(a) 1,200 sq. m; (b)* 2,400 sq. m; (c) 2,500 sq. m;

(d) 10,000 sq. m; (e) none of these.

39. If $1 + 2 + 3 + \cdots + k = n^2$ where n is less than 100, then k

(a) equals 1; (b) equals 1 or 8; (c) equals 8; (d)* equals 1, 8 or 49; (e) can take more than three (3) values.

40. Let f be a function defined on the real line such that

$$f(x) = \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{if } x \in Q^c. \end{cases}$$

Then f is

(a) continuous; (b) continuous only on Q; (c) continuous only on Q^c; (d) continuous only at some points in Q^c; (e)* none of the above.

Paper 2

Monday, 2 June 1975 Time 2.00 p.m. - 5.00 p.m.

1. Let f be a function given by $y = f(x)$. We say that f is identically zero if $y = 0$ for all given values of x.

Suppose that f is a function which is not identically zero, and that f satisfies

$$f(x) + f(z) = 2f(\tfrac{x+z}{2})f(\tfrac{x-z}{2}).$$

Prove: (i) $f(0) = 1$,

(ii) $f(2x) = 2f(x)^2 - 1$.

Give an example of such a function.

2. Let C be a right circular cone with height $\sqrt{1 - r^2}$ and base radius r such that $r < \frac{1}{2}$. A spider crawling on the curved surface of C wishes to take a turn round the cone and get back to where it was. If it starts from a point P at a distance x from the vertex, what is the shortest possible distance it can travel ? Suppose that, in taking the turn, the spider must pass through the point on the circumference of the base which is furthest away from P. Show that the shortest possible distance it can travel is $2\sqrt{1 - 2x\cos(\pi r) + x^2}$.

What happens if $r \geqslant \frac{1}{2}$?

3. (a) Find $\int \frac{\sqrt{a^2 - x^2}}{x^4}\, dx$,

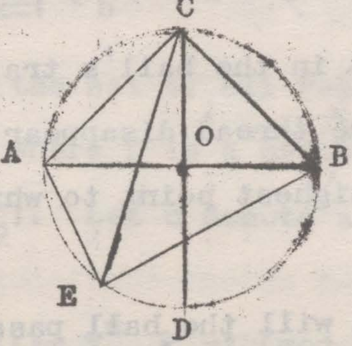
(b) If $f_n(x) = \frac{1}{n!}\int_0^x t^n e^{-t}\,dt$, prove that

$$f_n(x) - f_{n-1}(x) = -(x^n e^{-x})/n! \quad .$$

Hence, or otherwise, evaluate the integral

$$\int_1^2 t^3 e^{-t}\,dt.$$

4. (a) If AB and CD are perpendicular diameters of a circle, prove that the area of any inscribed quadrilateral ACBE (see Figure) is equal to $\frac{1}{2}\,\overline{CE}^2$.



(b) Prove that the sum of the three perpendiculars from a point inside an equilateral triangle to the sides is equal to the altitude of the triangle .

5. Let a_n be the probability that n independent trials with success probability p result in an even number of successes. Express a_n in terms of p and a_{n-1}. Show that $a_n = \frac{1}{2}\{1 + (1-2p)^n\}$.

6. Let a, b, c be odd integers. Prove that the roots of the equation

$$ax^2 + bx + c = 0$$

cannot be rational numbers.

7. A right circular cone is circumscribed to a sphere of

radius a, with the base of the cone touching the sphere. Find an expression for the volume of the cone in terms of a and the semi-vertical angle of the cone, and show that when the volume of the cone is a minimum it is twice the volume of the sphere.

8. A small ball is suspended from a point A by a weightless thread of length l. A nail is driven into the wall at a distance $l/2$ vertically below A. The ball is drawn aside so that the thread takes up a horizontal position and then released.

(a) At what point in the ball's trajectory will the tension of the thread disappear ?

(b) What is the highest point to which the ball will rise ?

(c) At what point will the ball pass through the vertical line passing through the point of suspension ?

9. Three uniform smooth cylinders, each of radius a and weight W, have their axes horizontal and parallel. Two of them lie on a smooth horizontal plane, not touching each other. The third lies symmetrically on top of them, not touching the plane. Equilibrium is preserved by a light endless string, which passes around them. Prove that the length $2\pi a$ of string is in contact with the cylinders and hence prove that, if the length of the string is $2\pi a + l$, then

$$6a < l < 8a.$$

Prove that the tension in the string is

$$\tfrac{1}{2} W (l - 4a)/\sqrt{l(8a - l)} .$$

10^+. (a) Let $[x]$ denote the greatest integer not exceeding the real number x. For example, $[3.9] = 3$, $[-1.14] = -2$ and $[\sqrt{2}] = 1$. Show that, if x is an integer and y not an integer, then $[x - y] = x - 1 - [y]$. Does the equality hold if x and y are both integers ?

(b) Let a and b be two positive integers and (a,b) denote the greatest common factor of a and b. Show that the only values of k for which ka/b is an integer are the multiples of b' where $b = (a,b)b'$. Further show that

$$ab - 2\sum_{k=1}^{b-1} \left[\frac{ka}{b}\right] = a + b - (a,b).$$

11. Let $\mathbb{Z}_p$ denote the set of all residue classes of integers modulo p, where p is a prime number. Write $\mathbb{Z}_p = \{x_1, x_2, \dots, x_p\}$. Let u denote a positive integer. Show that

(i) $x_1^u + x_2^u + \cdots + x_p^u \equiv -1 \pmod{p}$, if u is a multiple of $p - 1$.

(ii) $x_1^u + x_2^u + \cdots + x_p^u \equiv 0 \pmod{p}$, if u is not a multiple of $p - 1$.

(Hint: It is well-known that the set of all non-zero residue classes of $\mathbb{Z}_p$ forms a cyclic group under multiplication.)

$^+$ The following two problems are considered more difficult and carry 15 points each in the marking. The other problems carry 10 points each.

1. By substituting $x = z = 0$ into the equation given, one gets $f(0) = (f(0))^2$, so that $f(0) = 0$ or $f(0) = 1$. If $f(0) = 0$, then $2f(x) = 2f(x) \cdot f(0) = 0$, implying $f(x) = 0$ for all x. Since f is <u>not</u> identically zero, one concludes that $f(0) = 1$.

2.

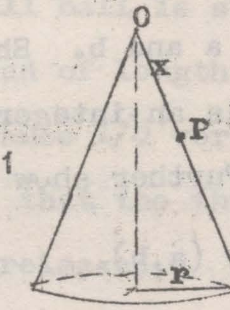


Fig. 1

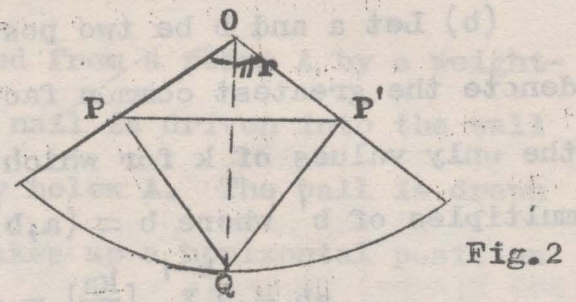


Fig. 2

Slice the cone along the straight line OP (see Fig. 1) to get Fig. 2. P' is the point such that $\overline{OP'} = \overline{OP}$. The shortest path to travel is therefore $\overline{PP'}$ which is equal to $2x \sin \pi r$. If the spider must pass through the point on the circumference of the base which is furthest away from P, then the shortest distance is $PQ + P'Q$ which is $2\sqrt{1 - 2x\cos\pi r + x^2}$, using the cosine rule of a triangle.

If $r \geqslant \frac{1}{2}$, then POP' forms an angle greater than or equal to 180°, so that the shortest possible distance will be $2x$. The shortest distance required in the other case, however, remains the same (see Fig. 3).

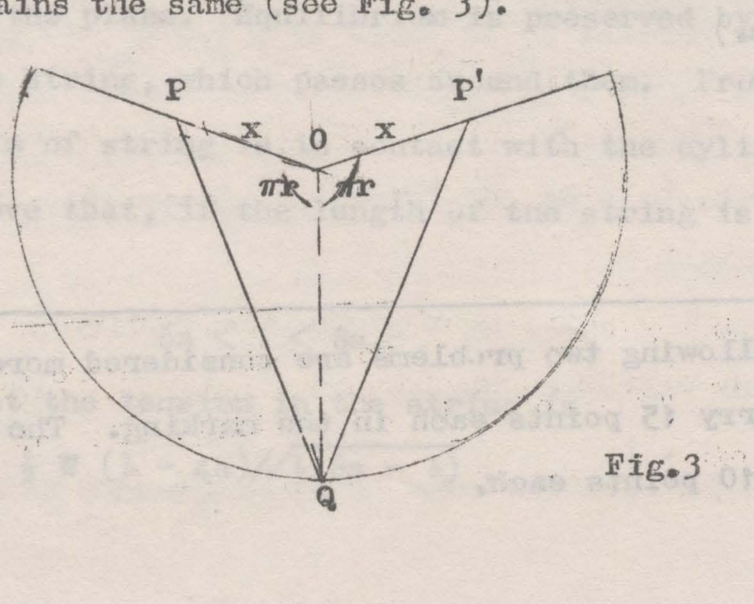


Fig. 3

Complete solutions were given by Chan Hock Chuan, Teo Siong Khow, Chua Yong Heng, and Chiew Tuan Chiong.

4. Let AR and BS intersect EC perpendicularly at points R and S respectively. Then Δ ARC is congruent to Δ BCS, so that $\overline{BS} = \overline{RC}$. Now Δ AER is isosceles, so that $\overline{AR} = \overline{RE}$. Hence $\overline{BS} + \overline{AR} = \overline{ER} + \overline{RC} = \overline{EC}$. But the quadrilateral ACBE is equal in area to area ΔAEC + area ΔBEC. The area of the sum of these two triangles is $\frac{1}{2}(\overline{AR} + \overline{BS})\cdot\overline{EC} = \frac{1}{2}\overline{EC}^2$.

Complete solutions were given by Wong Chee Teck, Eugel Yeo, Stephen Low Siang Kwang, Teo Siong Khow, Chew Tuan Chiong, Tay Yong Chiang, Wong Yue Kah, Wong Kwok Kwong, Tan Chee Mun and Lee Soon Pue.

5. Consider n independent trials. Let A be the event that the first trial is successful, B the event that there are an odd number of successes after the first trial, and C the event that there are an even number of successes after the first trial. Then

$$a_n = P(A \cap B) + P(A^c \cap C),$$

which by independence is equal to

$$\begin{aligned} &P(A)\cdot P(B) + P(A^c)\cdot P(C) \\ &= p(1 - a_{n-1}) + (1 - p)a_{n-1} \\ &= p + (1 - 2p)a_{n-1}. \end{aligned}$$

Solving the recursive equation, we obtain

$$a_n = p + (1 - 2p)p + (1 - 2p)^2p + \cdots + (1 - 2p)^{n-2}p + (1 - 2p)^{n-1}a_1,$$

$a_1 = 1 - p$ being the probability of obtaining an even number of, and therefore zero, successes. Hence

$$a_n = p\cdot\sum_{k=0}^{n-1}(1 - 2p)^k + (1 - 2p)^n = \tfrac{1}{2}\{1 + (1 - 2p)^n\}.$$

6. Suppose that $ax^2 + bx + c = 0$ has a rational root p/q with p and q relatively prime. Then

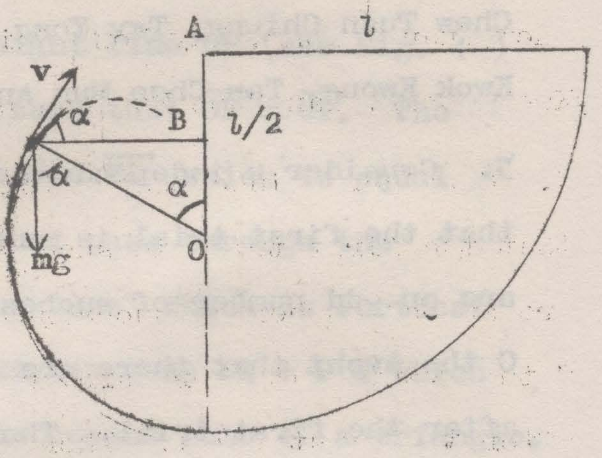
$$a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0$$

so that $$ap^2 + bpq + cq^2 = 0.$$

Case (i). If both p and q are odd then $ap^2 + bpq + cq^2$ is also odd and therefore cannot be zero.

Case (ii). If one of p and q is even, then two of the terms ap^2, bpq, cq^2 are even and the third term is odd. This leads to the same contradiction as in case (i).

Complete solutions were given by Tsai Wei Kang and Siew Chee Kin.



8. The ball first describes the quadrant of the circle of radius l. After touching the nail O, the ball describes an arc of a circle of radius $l/2$. Tension is zero when centripedal force balances the effect of weight. Let M be the position of the ball at which tension is zero, m the mass of the ball. Then $mg \cos \alpha$ is the component of the weight along the thread. Velocity acquired at M, denoted v, then satisfies

$$v^2 = 2gh = 2g\cdot\left(\frac{l}{2} - \frac{l}{2}\cos\alpha\right) = gl(1 - \cos\alpha).$$

Thus $\frac{mv^2}{l/2} = 2mg(1 - \cos\alpha)$, so that at point M,

$$mg \cos\alpha = 2mg(1 - \cos\alpha), \text{ giving } \cos\alpha = \frac{2}{3}.$$

This defines the position of M. The ball continues its path as a body thrown at an angle α to the horizontal with initial velocity $v = \sqrt{2gh} = \sqrt{lg/3}$. The highest point H is attained when $2gH = (v\sin\alpha)^2$, giving $H = 5l/54$.

The segment MB is calculated to be equal to $\sqrt{\frac{5}{6}}\, l$.

To travel this distance, the time t required is $(\sqrt{15}/4)(\sqrt{l/g})$. Then the vertical distance travelled by the ball within this time is $vt\sin\alpha - \frac{gt^2}{2} = -\frac{5l}{96}$, i.e. $\frac{5l}{96}$ below B.

Complete solutions of this problem were given by Ronald Tan Hee Huan, Chew Cheah Boon and Terence Siew Chee Kin.

10. (a) If y is not an integer, then $y = [y] + \alpha =$ $[y] + 1 - (1 - \alpha)$ where $0 < \alpha < 1$ and so $0 < (1 - \alpha) < 1$. Therefore if x is an integer and y not an integer, we have $[x - y] = [x - 1 - [y] + (1 - \alpha)] = x - 1 - [y]$.

If both x and y are integers then $[x - y] = x - y = x - [y]$.

(b) Since $a = (a,b)a'$ and $b = (a,b)b'$, we have

$$\frac{ka}{b} = \frac{ka'}{b'} \qquad \text{where } (a',b') = 1.$$

Therefore ka/b is an integer if and only if b' divides ka' if and only if b' divides k, i.e. k is a multiple of b', by virtue of $(a',b') = 1$.

To prove the identity, write $[\frac{ka}{b}] = [a - \frac{(b-k)a}{b}]$. For $1 \leqslant k \leqslant b-1$, there are exactly $(a,b) - 1$ many k's for which k is a multiple of b' and so $\frac{ka}{b}$ is an integer. Therefore $[\frac{ka}{b}] = a - 1 - [\frac{(b-k)a}{b}]$ except for $(a,b) - 1$ many k's for which $[\frac{ka}{b}] = a - [\frac{(b-k)a}{b}]$. So

$$\sum_{k=1}^{b-1} [\tfrac{ka}{b}] = \sum_{k=1}^{b-1} \{a - 1 - [\tfrac{(b-k)a}{b}]\} + (a,b) - 1$$

$$= (a-1)(b-1) - \sum_{k=1}^{b-1} \left[\frac{(b-k)a}{b}\right] + (a,b) - 1$$

$$= ab - a - b + (a,b) - \sum_{k=1}^{b-1} \left[\frac{ka}{b}\right].$$

This proves the identity.

11\. (i) Since the set of all non-zero residue classes of $\mathbb{Z}_p$ forms a group under multiplication, we have

$$x_n^{p-1} \equiv 1 \pmod{p},$$

where $x_n \not\equiv 0 \pmod{p}$ and $x_n \in \mathbb{Z}_p$. Let $u = (p-1)v$. Then

$$x_n^u \equiv 1^v \equiv 1 \pmod{p}.$$

$$\begin{aligned} \text{Hence } x_1^u + x_2^u + \cdots + x_p^u &\equiv 0 + 1 + 1 + \cdots + 1 \\ &\equiv p - 1 \pmod{p} \\ &\equiv -1 \pmod{p}. \end{aligned}$$

(ii) First note that the set of all non-zero residue classes of $\mathbb{Z}_p$ forms a cyclic group. Let x_m be a generator of this group. Then if u is not a multiple of $p-1$, p is not a factor of $x_m^u - 1$, so that

$$(*) \qquad x_m^u \not\equiv 1 \pmod{p}.$$

Write $S = x_1^u + \cdots + x_p^u$. Then $x_m^u S = x_m^u x_1^u + \cdots + x_m^u x_p^u = (x_m x_1)^u + \cdots + (x_m x_p)^u$. But $x_m x_1, \ldots, x_m x_p$ are all the distinct elements of $\mathbb{Z}_p$ (for if not, one has $x_m x_i \equiv x_m x_j \pmod{p}$ for some $i \neq j$. Multiplying by the inverse of x_m on both sides of this equation gives $x_i \equiv x_j \pmod{p}$, which is a contradiction). Therefore $x_m x_1, \cdots, x_m x_p$ is just a rearrangement of $x_1, \ldots, x_p$. Hence

$$x_m^u S \equiv S \pmod{p},$$

i.e. $(x_m^u - 1) S \equiv 0 \pmod{p}$, implying that p is a factor of $(x_m^u - 1) S$. But p is a prime number, so p divides $x_m^u - 1$ or S. Because of $(*)$, p has to divide S, i.e.

$$S \equiv 0 \pmod{p} \text{ as required.}$$

There were five attempts to solve this problem. Two students - Hee Juay Guan and Teo Siong Khow - obtained partial results, but no one received full credit for part (ii). The small number of students attempting this problem is probably due to the fact that not too many schools use the new syllabus. In fact the five attempts were from students of five different schools, suggesting that they were individual efforts.

A Letter to the Editor

...I suggest that there should be a column for queries by practising school teachers. Through this column they may ask for

(i) methods on how to teach a certain topic;

(ii) solutions to a problem which the teacher has failed to solve.

Then readers of the Medley may come to the rescue. If there is such a column, a copy of the Medley should be sent to the Principals of Secondary Schools. In this way we hope that more teachers will participate in the activities of the Society.

I should like to add a short note to the book review by Ho Soo Thong (this Medley, Vol.3, No.1) of 'Statistical Mathematics' by R. Loveday. The book's answer to Miscellaneous Exercise No. 19, page 86, is wrong. In other words,

$$n_1 + 2n_2$$

$$n_1 + 2n_2$$

$$[e^{-(n_1+2n_2)}(n_1+2n_2)^5]/5!$$