WHAT IS MATHEMATICAL LOGIC?

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Bertrand Russell is dead but his mathematical logic lives on. Indeed, mathematical logic, a subject which few had even heard of ten or twenty years ago is now taught to undergraduate mathematicians in most good universities.

What is mathematical logic? We know what mathematics is: it is the discipline of calculating (in some sense). Roughly speaking mathematical logic is the discipline of calculating with formal languages.

Logic is generally, and quite rightly, regarded as an area in which philosophers work. Let me present a simile. As in physics mathematical equations are used to mirror physical situations and physical phenomena, so in mathematical logic mathematical symbolism is used to treat philosophical arguments.

The problems with which mathematical logic is concerned have often arisen directly as the result of philosophical researches into the foundations of mathematics. Thus Bertrand Russell's paradox showed that the notion of an arbitrary collection, or set, was not clear. And it is in this area of the classification of the notion of set that much of the most important current work in mathematical logic is being done.

The symbolism which was introduced, principally around the beginning of this century, made philosophical arguments susceptible to mathematical attack. The very symbolism could be treated as a mathematical structure and could be subjected to analysis. This is quite similar to the way children analyse patterns of Cuisenaire rods in primary schools. The philosophical arguments correspond to the actual configurations of rods, mathematical logic to the analysis of patterns in terms of addition and subtraction. The rules for adding and subtracting are very simple and it is certainly true that the rules for combining symbols in mathematical logic are deceptively simple too. I say "deceptively simple" because the simple framework is very much akin to basic parts of an electronic computer; but just as computers now carry out exceedingly complicated operations, so it turned out that the "deceptively simple framework" was sufficient for the presentation of all known mathematics. This was Russell's great contribution.

That there should be a significant connection between computers and mathematical logic is no accident. The famous English logician, Alan Turing, was deeply involved in both computers and mathematical logic. He invented a class of theoretical computers, Turing machines, and it is generally maintained that any function whose value can be calculated mechanically can be calculated by such a Turing machine. In order to specify any Turing machine one only has to list the simple operations it performs. These are all purely mechanical and can be described in finite terms.

The study of Turing machines constitutes an area of mathematical logic, the theory of recursive functions. Althouth it would be misleading to say that the theory of recursive functions has dominated the theory of computers, it has certainly had a big impact on it.

Recursive functions also play a vital role in the proof of the most well-known theorem of mathematical logic: Gödel's incompleteness theorem. Here is a brief outline of this theorem.

Suppose we write out axioms and rules of inference for arithmetic in much the same way as, a long time ago, Euclid did for geometry. Then Gödel's theorem is essentially that, however we set up a finite collection of such axioms and rules, there will always be some statement of arithmetic which, though true, cannot be established from our axioms. Gödel obtained this result by showing that recursive (that is, mechanically computable) functions and the formulae of arithmetic could be described by giving them numbers in such a way that all the statements could be numbered off. Then he showed that amongst these statements there is one which actually says: "The statement with number n does not have a proof" where this sentence is itself the statement with number n. Now if there is a proof of the statement "The statement with number n does not have a proof" then this statement is false. So this statement cannot have a proof. If it does not have a proof, then what it says, which we may paraphrase as "This sentence does not have a proof" is clearly true.

It is only fair to add that there is quite a lot of complicated work to do in order to fill in all the gaps in the sketch of the proof of Gödel's incompleteness theorem which I have just given. But I think one point is clear, that there is a significant difference between 'provability' (in a formal language) and 'truth' when one is dealing with arithmetic.

In the case of the ordinary logic of affirmation and negation truth and provability (from a suitable set of axioms) coincide. This result was obtained in the first half of this century.

The original impetus for all this work came from problems that arose in work on Cantor's set theory. Part of this set theory is now being taught in primary schools. The problems encountered can, to a large extent, be avoided by setting up a system of axioms and rules of inference in the spirit of Euclid. These axioms are subject to Gödel's incompleteness theorem which we mentioned above, since arithmetic can be done within set theory. But although the axioms for set theory which have so far been proposed are insufficient for us to derive all true statements, they do provide a perfectly adequate framework for almost all of our known mathematics. Of course, in order to do their job properly, these axioms must not contradict each other. This is the problem of the 'consistency' of the axioms. Firstly, even from a purely mathematical point of view it may be difficult to see whether the axioms are consistent and secondly, even if they are consistent, the techniques required to establish consistency may be more problematical than the set theory itself. Fortunately for the working mathematician it is clear that the axioms of set theory we do use are as reliable as anything else! A well-known logician has remarked that there are not likely to be any bridges falling down because of a lack of firm foundations of mathematics.

This brings us to the present day and I am often asked: What do mathematical logicians do? How can anyone do research in mathematics? The second question is fairly easy to deal with. I said earlier that mathematical techniques are applied to the logical problems thrown up by the philosophers. There are also problems generated by research already begun and there is the development of particular areas for their intrinsic interest. I mention recursive functions in this context for there is a continuing connection between recursive function theory and its techniques and the theory of computation. The logician leads an exploration into uncharted areas of thought, the computer theorist, in addition to following up his own ideas, seizes on and develops in his own way, the creations of the recursive function theorist.

And finally the other question: What do mathematical logicians do? We do research in an area of mathematics. Often we apply mathematics to logic, other times we apply logic, or more particularly techniques of logic, to mathematics to obtain purely mathematical results.

Some Suggestions for further reading

J. N. Crossley and Others: What is Mathematical Logic? Oxford University Press 1972 (paperback)

H. Enderton: A Mathematical Introduction to Logic Academic Press 1972.

S. C. Kleene: Mathematical Logic. Wiley 1967.

R. C. Lyndon: Notes on Logic, van Nostrand lecture notes (paperback). 1966.

E. Mendelson: Introduction to Mathematical Logic, van Nostrand 1964.

(Professor Crossley's article is a written version of a talk delivered to the Society on 19 July 1974)

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NOTES ON MATHEMATICIANS

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Introduction

Pure mathematics splits fourfold into (1) Mathematical Logic and the Foundations of Mathematics, (2) Algebra and the Theory of Numbers, (3) Analysis and Geometry and (4) Topology. Of these subjects, the Theory of Numbers and Geometry undoubtedly have the longest history, dating back to some two thousand years. Other subjects developed much later, with Analysis and Algebra around the same time in the eighteenth century, Topology in the late nineteenth century, and finally Logic at the turn of this century.

Over the years these subjects evolve and intermingle with one another, while at the same time they expand further and further at a fascinating rate. Today there is hardly any mathematician who can claim to be a universalist. Not so fifty years ago! David Hilbert eminently qualified as one, his contributions having covered every subject in pure and applied mathematics. Before him there were Poincaré and Gauss. But still there have been very few.

How did the subjects evolve? Who were the prime movers, the great contributors? In this series of notes we shall introduce men who in our opinion develop mathematics into what it is today, and because of whom mathematics is never the same again.

A great mathematical work is like a great work of art. It is the result of a complete devotion to the subject, a highest degree of concentration of the mind and an exploitation to the limit of man's faculty of thought, by which intricate and ingenious logical arguments are conceived to take care of alll the difficulties involved in the successful completion of a work.

Therefore it takes years to accomplish such a feat. The satisfaction lies in seeing a rock gradually getting carved into