PROBLEMS AND SOLUTIONS

A $5 book-voucher prize will be awarded to the Junior Member who submits the best solution to any starred problem before 30 June 1975. More than one solution may be submitted. In the event of equally good solutions being received, the prize or prizes will be awarded to the solution or solutions sent with the earliest postmark. In the case of identical postmarks, the winning solution will be decided by ballot.

Problems or solutions should be sent to Dr. Y.K. Leong, Department of Mathematics, University of Singapore, Singapore 10. If a solution to a submitted problem is known, please include the solution.

Correction to P6/74. The Proposer and Dr. Louis Chen have pointed out typographical and mathematical errors in the statement of P6/74 (this "Medley", Vol. 2, No.2, p.13). The Editor apologises for the errors. The correct statement of P6/74 is given below.

If x,y,k are positive numbers with $x \neq y$, $k > 1$, prove that

(i) $$\frac{x}{x+ky} + \frac{y}{y+kx} > \frac{2}{k+1},$$

(ii) $$\frac{x}{y+kx} + \frac{y}{x+ky} < \frac{2}{k+1}.$$

(Leonard Y. H. Yap)

P1/75. If n and p are positive integers, show that $(np)!/(n!(p!)^n)$ is an integer.

(Chan Sing Chun)

*P2/75. If A and B are any two n x n matrices, prove that $AB - BA \neq I_n$, where I_n is the n x n identity matrix.

(P. H. Diananda)

*P3/75. Let a,b,c,d,e be any real numbers and $d \neq 0$. Prove that the equation

$$x^3 + (a+b+c)x^2 + (ab+bc+ca-d^2)x + e = 0$$

has at least two distinct roots.

(H.N. Ng)

(Hint: $a^2+b^2+c^2-ab-bc-ca \geq 0$.)

Solutions to P7 - P10/74

P7/74. If p,q,r are distinct primes, show that $p^{1/3}$, $q^{1/3}$, $r^{1/3}$ do not form an arithmetic progression in any order.

Solution.

Suppose that $p^{1/3}$, $q^{1/3}$, $r^{1/3}$ form an A.P. in that order. Then

$$q^{1/3} = \tfrac{1}{2}(p^{1/3} + r^{1/3}).$$

Taking cubes on both sides, we have

$$8q = p+r+3(pr)^{1/3}(p^{1/3} + r^{1/3}),$$

or $6(pqr)^{1/3} = 8q - p - r = m$, say.

Hence $6^3(pqr)=m^3$, and so 6 must divide m. Writing m=6n for some integer n, we get $pqr=n^3$. It follows that p divides n, i.e. n=pk for some integer k. Thus $qr=p^2k^3$, which means that p must divide q or r. This is impossible since p,q,r are distinct primes.

Note that the result is true as long as pqr is not a perfect cube.

P8/74. Without using tables, show that

$$(1.3.5...99)/(2.4.6...100) < 1/10.$$

Solution by Chan Sing Chun

Let a=(1.3.5...99)/(2.4.6...100) and b=(2.4.6...100)/(3.5.7...101). We have $1/2 < 2/3$, $3/4 < 4/5, \ldots, (2n-1)/(2n) < (2n)/(2n+1)$, ..., $99/100 < 100/101$. Multiplying these inequalities together, we have $a < b$. Hence

$$a^2 < ab = 1/101 < 1/100,$$

and so $a < 1/10$.

P9/74. Show that for all positive integers m and n,

$$\binom{m+0}{0} + \binom{m+1}{0} + \ldots + \binom{m+n}{n} = \binom{m+n+1}{n},$$

where $\binom{m+i}{i}$ is the binomial coefficient $(m+i)!/m!i!$.

Solution by Chan Sing Chun.

For i=0,1,2,...; $\binom{m+i}{i}$ is the coefficient of x^m in the expansion of $(1-x)^{-(i+1)}$. Hence the given sum is equal to the coefficient of x^m in the expansion of

$$(1-x)^{-1} + (1-x)^{-2} + \ldots + (1-x)^{-(n+1)}$$

$$= (1-x)^{-1}\left\{1 - (1-x)^{-(n+1)}\right\} / \left\{1 - (1-x)^{-1}\right\}$$

$$= -x^{-1} + x^{-1}(1-x)^{-(n+1)}.$$

The coefficient of x^{m+1} in the expansion of $(1-x)^{-(n+1)}$ is precisely $\binom{m+n+1}{n}$.

Alternative solutions.

(1) Use induction on n. The result is easily verified when n=1. So assume that the result holds for $n=k \geq 1$. Hence

$$\binom{m+0}{0} + \binom{m+1}{1} + \ldots + \binom{m+k+1}{k+1} = \binom{m+k+1}{k} + \binom{m+k+1}{k+1} = \binom{m+k+2}{k+1}.$$

This shows that the result is again true when n=k+1.

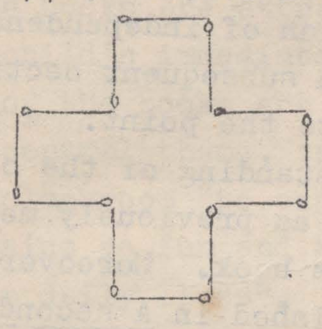
(2) Write $u_i = \binom{m+i}{i}$. Define $v_i = \binom{m+i}{i-1}$ for i=1,2,... .

It is easily checked that $u_i=v_{i+1}-v_i$, $i=1,2,\ldots,n$. Adding these n equations, we have

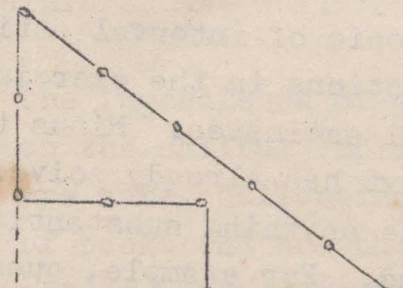
$$u_1+u_2+\ldots+u_n = v_{n+1}-v_1 .$$

Since $v_1=1=u_0$, the result is proved.

P10/74. Twelve matches, each of unit length, form the figure of a cross as shown. Rearrange the matches in such a way as to cover an area of four square units.



Solution.



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BOOK REVIEWS

Books for review should be sent to Mr. Lim Chee Lin, Hwa Chong Junior College, Bukit Timah Road, Singapore 10.

Statistical mathematics, by Robert Loveday, Cambridge University Press, Cambridge, 1973, viii + 104 pp, £1.10.

This book is a companion volume of the author's A second course in statistics and is written with two aims in mind: to present a mathematical approach to statistics and provide guidance to students preparing for Advanced Level Examinations. With respect to these aims, the book serves its purpose. The proofs are not only convincing in their mathematical treatment, but also challenging to the enthusiastic reader. The abundance of past examination questions serves excellently the second objective.

Below are some specific comments on each chapter.

Chapter 1. A second course in statistics does not contain a chapter on continuous distributions. It is justifiable, therefore, that Chapter 1 should include some illustrative examples